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ANALYTICAL STUDY OF SHIMMY OF AIRPLANE WHEELS

By Christian Bourcier de Carbon

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By Christian Bourcier de Carbon

SUMMARY

The problem of shimmy of airplane wheels is particularly important in the case of the tricycle nose wheel.

There is no rational theory of this dangerous phenomenon, although some tests have been made in the U. S. A. and in Germany.

The present report deals with a rather simple theory that agrees with the known tests and enables dimensions of the land-gear to be computed so as to avoid shimmy without resorting to dampers. Tests with a full-scale landing gear are described.

I. THE PROBLEM OF SHIMMY

When a wheel fitted with a tire is designed to pivot freely about a vertical pivot AA' (fig. 1) and this pivot is given a horizontal forward motion while the wheel is made to roll on the ground, it happens that it spontaneously assumes a self-sustained oscillating motion about the pivot AA'. This phenomenon popularly termed shimmy or wiggle is frequently observed on the tail wheel of a conventional airplane and on the nose wheel of tricycle landing gears. In the latter case it can become extremely violent and consequently very annoying, and occasionally induce failure of the landing gear. In fact, it has been called the "bête noire" (bugbear) of the tricycle landing gear: "the problem of nose wheel shimmy, long the bugaboo of tricycle landing gear . . . ." (Aerodigest, March 15, 1944, p. 134).

The Americans, promoters of the tricycle landing gear, have already made large-scale theoretical and experimental studies of this phenomenon, but as yet, there seems to be no complete and acceptable theory. The writer proposes to supply this theory in the present report. The suggested explanation involves only the elementary mechanical properties of the tires which are all well defined and easily measurable.

*"Étude Théorique du Shimmy des Roues d'Avion", Office National d'Études et de Recherches Aéronautiques, Publication No. 7, 1948.

Before any particular hypothesis, it is, in fact, quite evident that shimmy can have no cause other than the reactions exerted by the ground on the wheel-reactions transmitted by the tire. These reactions can only be contact reactions, due to the adherence of the tire; they are necessarily reduced to a force and to a torque. To analyze these reactions more correctly it is necessary, first of all, to form the theory of the elementary mechanical properties of the tire.

II. THE MECHANICS OF THE TIRE

A tire is not merely an elastic element possessing vertical elasticity which is essentially its reason for existence, that is, a simple vertical spring placed between ground and wheel; it is a complex mechanical unit having a certain number of other elementary mechanical properties. For the ensuing theory the four following fundamental factors are considered:

1. Lateral Elasticity
2. Torsional Elasticity
3. Drifting
4. Turning

1. Lateral Elasticity

At rest, under the action of a relatively moderate force F, applied parallel to the wheel, that is, perpendicular to its plane, the wheel is displaced with respect to the ground, and through the simple effect of the deformation of the tire, by an amount Δ proportional and parallel to the force F. This displacement is due, as shown in figure 2, to the simple deformation of the tire near the surface of contact with the ground, without any modification of that contact surface as long as the force F does not exceed the limit of adherence.

In the formula

$$\Delta = TF$$

T is the coefficient of lateral elasticity of the tire.

2. Torsional Elasticity

At rest, under the action of a relatively moderate moment torque $\underline{M}$ and vertical axis applied to the wheel, the latter is subjected to a rotation of angle α proportional to the moment $\underline{M}$. As in the preceding case, the mechanism of this rotation is easy to understand. It results from the deformation of the tire near the surface of contact with the ground, this surface itself remaining unaltered as a result of the adherence.

In the formula

$$\alpha = S\underline{M}$$

S is called the coefficient of torsional elasticity of the tire.

3. Drifting

Under the action of the side force F a wheel mounted with a tire is, according to the foregoing, subjected, first, to the lateral displacement $\Delta = TF$, that is, its center is shifted from O to O' and its plane from P to P' (fig. 3). In addition, if the wheel is made to roll, it is seen that its track is no longer contained in the plane of the wheel, it moves without sliding along a straight line O'M, forming an angle δ proportional to the force F.

In the formula

$$\delta = DF$$

D is termed the drift coefficient of the tire. This phenomenon is in effect, a drift comparable to the drift of a ship in side wind. It is particularly evident on a bicycle, or better yet, on a motorcycle with an insufficiently inflated front wheel. It is this deflection of the rubber tires that enables mothers to steer with ease baby carriages with non-swiveling wheels without being forced to make the wheels skid on the ground or to raise two of the wheels in order to turn the carriage.

A remarkable thing, this drift is produced without the least sliding of the tire over the ground. It, as well as the other properties of the tire, can be explained by a very simple diagram. Supposing the tire is equivalent to an infinite number of small coiled springs mounted radially on the periphery of the wheel and endowed with elasticity in the radial and the lateral directions (fig. 4). This evidently represents the vertical and lateral elasticity of the tire, as well as the torsional elasticity. It also explains the drift. In effect:

Suppose that the wheel is at rest; a certain number of springs n are in contact with the ground, as outlined in figure 5. Now, when a side force F is applied to the wheel, it first is subject to a lateral displacement $\Delta = TF$ as a result of the simple lateral deformation of the n springs in contact with the ground without the least sliding of their contacts with the ground. As a consequence each of these springs assumes a certain lateral tension, the sum of these n tensions balancing the force F. Now the wheel is made to roll. Obviously, on account of the displacement Δ the contact of the spring $n + 1$ with the ground is no longer in the alignment of the first n's, but displaced laterally by an amount equal to Δ. On the other hand, at the moment that the spring $n + 1$ makes contact with the ground, the spring 1 loses this contact; but this spring has a certain lateral tension which suddenly stops as soon as the spring leaves the ground, while at the instant of making contact with the ground the spring $n + 1$ has as yet no tension. The sum of the reactions of the ground is therefore diminished by the tension of spring 1 and it no longer balances the force F. To recover this reduction the wheel is again subjected to a slight lateral displacement so that the point $n + 2$ is again displaced or forced off laterally with respect to the point $n + 1$. The process is repeated every time one spring leaves the ground and another makes contact. It is seen that the wheel rapidly assumes the state of a lateral displacement proportional to the path traveled in the direction of rolling of the wheel. The normal state can be considered as reached as soon as the point n has left the ground, that is, as soon as the wheel has covered a distance equal to the length l of the surface of contact. This length being slight, it all happens as if that state was instantaneous. Since each one of the elementary displacements involved in this analysis is proportional to F, the same applies to their sum, hence the formula

$$\text{drift} = DFx$$

x = distance traveled in direction of rolling, and consequently

$$\text{angle of drift} \quad \delta = DF$$

Thus the drift is explained by the natural elasticity of the tire without involving the least effect of skidding or sliding of the tire over the ground. This phenomenon is almost the same as the longitudinal creep of transmission belts; it is fully comparable with pseudo-slipping in which the drive wheels of a vehicle make a number of revolutions greater than that resulting from the distance traveled by the vehicle and greater than that of nondriving wheels of the same diameter carrying the same load. Pseudo-slipping which has been the subject of many experimental studies for locomotives, is explainable in the same manner without involving real slipping between wheel and rail; it results from elastic deformations of the steel near the surface of contact and its simplified theory is the same as that of drift. Pseudo-slipping is a longitudinal drift and the drift a lateral pseudo-slipping. Likewise, braking also produces negative pseudo-slipping.

Within certain limits the drifting of a wheel can be compared to the deformation of a solid body. The displacement Δ plays a part similar to that of the reversible elastic deformation, and drift DFx similar to that of the irreversible permanent deformation arising from a viscous creep, the path covered x in the drifting playing the part of the time in creep.

The torque accompanying the drift.- The preceding analysis indicates that the drift is necessarily accompanied by a very important mechanical phenomenon, namely, the displacement of the lateral reaction of the ground toward the rear of the wheel. Return to figure 4 and consider the wheel at the moment the spring $k + n$ comes in contact with the ground. On account of the drift the points $k + 1$, $k + 2$, . . . $k + n$, are alined, conformably as in figure 6, along a straight line forming an angle δ with the plane OP of the wheel. On the other hand, at the instant point $k + n$ makes contact with the ground, the plane of the wheel obviously passes through the point $k + n$, which is termed the head point of contact of the wheel with the ground. Since the tension is proportional to the distance between ground contact and wheel plane the result is a lateral tension of some of the n springs in contact with the ground proportional to the distance between ground contact and head point. This tension varies therefore from zero for head point $k + n$ to a maximum value for the tail point $k + 1$. The reaction F' of the ground, the resultant of these different tensions, therefore passes through the point G located at two-thirds of the line of contact from the head point. The ground reaction F' therefore does not pass through the center M of the line of contact. While parallel, equal and of opposite sign of F, the reaction F' is therefore not directly opposite to the force F which, passing through the projection O of the center of the wheel, passes also through M. The result is a torque $\underline{C}$ about the vertical axis of the wheel.

With ϵ denoting the distance MG and noting that the angle of drift δ is practically always small, this torque has the value

$$\underline{C} = \epsilon F$$

In the case of the wheel shown in figure 4 it is readily apparent that $\epsilon = MG = \frac{l}{6}$, where l = length of contact, that is, distance between head point and tail point. But it is fitting to note at the same time that, while the scheme is helpful for understanding the drift and the phenomenon of the accompanying torque, the tire is nevertheless a more complicated mechanism. The simplified scheme supposes that the small radial springs, equivalent to a tire, are independent. But owing to the continuity of the pneumatic tire, these springs must be considered strongly tied elastically to one another in such a way that the deformation of one of them also involves almost as much deformation of the two adjacent ones.

However, it can be shown that between ϵ and l the important relation

$$\frac{l}{6} < \epsilon < \frac{l}{2}$$

exists.

For the present, without stopping to demonstrate this formula, the following are assumed experimental facts:

(1) Every relatively moderate lateral force F applied to a tire in motion produces a drift proportional to the angle $\delta = DF$.

(2) This drift is accompanied by a couple $\underline{C} = \epsilon F$ tending to orient the wheel in the true direction of displacement, that is to reduce the angle of drift spontaneously.

The coefficient ϵ for a given load supported by the wheel is a characteristic constant of the tire which is termed length of displacement; and torque $F\epsilon$ the torque accompanying the drift.

4. Turning

Under the action of a couple of relatively moderate torque $\underline{M}$ about the vertical axis, applied to the center of the wheel, the latter is subjected, first, to a static rotation of angle α, as shown in the foregoing. But, when the wheel is made to roll, it no longer moves along a straight track, but along a circular trajectory of radius ρ, according to figure 7. The arc of circle AB is such that its curvature is proportional to the moment $\underline{M}$. Hence

$$\frac{1}{\rho} = R\underline{M}$$

that is, for a trajectory AB of length s, the wheel has turned through an angle

$$\beta = R\underline{M}s$$

R is termed the coefficient of turning of the tire.

To a certain degree the rotation of the wheel can be compared with the deformation of a solid body. The angle α then plays a part similar to that of the reversible elastic deformation, and angle β similar to that of the irreversible permanent deformation.

It is seen that the moment $\underline{M}$ necessary to make the wheel turn to a permanent angle β is inversely proportional to the traveled distance s. It is this property of turning that explains the ease of steering the wheels of an automobile in operation while turning the wheels of an automobile when standing still is much more difficult because it can only be achieved by an entirely different mechanism, namely, by exceeding the limit of adhesion of the tire with the ground. If this adhesion were perfect, that is, if no sliding were possible, the permanent turning of the wheel at rest would be impossible, whereas the sliding of the wheel plays no part in steering the wheel when rolling, as will be proved elsewhere in a more rigorous analysis of this phenomenon. As for the drift, turning can be explained on the basis of figure 4; the process is exactly the same; simply replace the lateral displacements by rotations. Turning is to the torsional elasticity what deviation is to the lateral elasticity. The reality of turning is assumed as an experimentally established fact. As regards the proportionality of rotation β to moment $\underline{M}$ (like the proportionality of drift to side force) from the moment the existence of such rotation is assumed the proportionality must be assumed proved from the following very general reasoning: The tire is a complex elastic system, whatever the effects of the forces, they must be proportional, at least as long as the forces do not exceed certain limits.

The properties of turning and drift in question indicate that a wheel fitted with a pneumatic tire, or more general, any wheel fitted with any tire has the important property of being able to roll while making a certain angle with its trajectory. This is the essential phenomenon which, as will be shown, underlies and basically explains the shimmy of pneumatic tires.

The five characteristic tire factors produced by the foregoing analysis are

(1) Coefficient of lateral elasticity T defined by the equation

$$\Delta = TF$$

(2) Coefficient of torsional elasticity S defined by equation

$$\alpha = S\underline{M}$$

(3) Coefficient of drift D defined by equation

$$\delta = DF$$

(4) Coefficient of turn R defined by equation

$$\beta = R\underline{M}s$$

(5) Length of displacement ϵ defined by equation

$$\underline{C} = \epsilon F$$

III. MATHEMATICAL THEORY OF ELEMENTARY SHIMMY

In order to obtain a clear picture of the mechanism of this phenomenon, the number of parameters are reduced by starting the study with what may be called elementary shimmy or shimmy with one degree of freedom, that is, the shimmy obtained when the spindle axis of the wheel is assumed to be impelled only by a straight, uniform horizontal motion. This is equivalent to assuming the pivot absolutely rigid and the mass of the airplane very great with respect to that of the wheel, so that the effects of wheel reactions on the trajectory of the caster axis can be disregarded. The position of the wheel with respect to the airplane is thus defined by one parameter, the angle θ of its plane with the axis of the airplane (fig. 8).

The study of elementary shimmy is divided in two parts.

In the first part is assumed that the reactions of torsion and turn are negligible compared with the reactions of lateral elasticity and drift, that is that the effects of torque $\underline{M}$ are negligible compared with those of force F, which is the same as assuming that the coefficient R is sufficiently great with respect to coefficients T and D. Later on it shall be shown that this is practically the case for ordinary tires. It produces a simplified theory of elementary shimmy. The second part contains the complete theory of elementary shimmy.

A. Simplified Theory of Elementary Shimmy

It is assumed that the pivot is vertical and the pivoting free, that is without restoring torque and without braking.

Let P denote the track on the ground of the pivoting axis at instant t, x the distance traveled by this point, O the projection of the wheel center, θ the angle of the line PO with the trajectory PX of the point P, that is with the axis of the airplane, AMGB the axis of symmetry of the contact surface of the tire with the ground, A being the head point, B the tail point, M the middle of AB and G the center of the ground reactions, that is the point so that $MG = \epsilon$, y and z the distances from PX of points M and O and finally that ψ is the angle of the line of contact AB with PX.

Considering now the place of point M on the ground whose tangent is the straight line AB, since the point M passes successively throught points B and A due to the rolling of the wheel. Hence the formula (fig. 8)

$$\frac{dy}{dx} = \psi$$

It is readily seen that the angle of drift δ is equal to $\theta + \psi$, and since $\theta = \frac{z}{a}$, hence

$$\delta = \frac{dy}{dx} + \frac{z}{a}$$

Now, if F is the resultant of the ground reactions, the preliminary study made on drift makes it possible to write

$$\delta = DF$$

hence the first equation reads

$$\frac{dy}{dx} + \frac{z}{a} = DF$$

On the other hand, the action of the lateral elasticity yields the second equation

$$z - y = TF$$

Lastly, the force F being applied at G, that is a distance $a + \epsilon$ from point P, produces a torque equal to $-F(a + \epsilon)$ with respect to this point. In other words, to the primary torque $-Fa$ the accompanying torque $-F\epsilon$ must be added. So, if I signifies the inertia of the whole oscillating system with respect to the pivot P, the third equation reads

$$I \frac{d^2\theta}{dt^2} = -F(a + \epsilon)$$

Noting that in the first equation

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{v}\frac{dy}{dt}$$

where v = speed of airplane, and that in the third equation

$$\theta = \frac{z}{a}$$

the system

$$\frac{1}{v}\frac{dy}{dt} + \frac{z}{a} = DF \tag{1}$$

$$z - y = TF \tag{2}$$

$$\frac{I}{a}\frac{d^2z}{dt^2} = -F(a + \epsilon) \tag{3}$$

is obtained. The elimination of F from equations (1) and (3) leaves

$$\frac{1}{v}\frac{dy}{dt} + \frac{z}{a} + \frac{ID}{a(a + \epsilon)}\frac{d^2z}{dt^2} = 0 \tag{4}$$

On the other hand, the elimination of F from equations (2) and (3) leaves

$$y = z + \frac{IT}{a(a + \epsilon)}\frac{d^2z}{dt^2} \tag{5}$$

Substitution of this value of y into the preceding equation finally gives the differential equation of the third order

$$\frac{IT}{v(a + \epsilon)}\frac{d^3z}{dt^3} + \frac{ID}{a + \epsilon}\frac{d^2z}{dt^2} + \frac{a}{v}\frac{dz}{dt} + z = 0 \tag{6}$$

which governs the motion of the wheel center. The general solution of this equation is

$$z = C_1e^{s_1t} + C_2e^{s_2t} + C_3e^{s_3t}$$

where C_1, C_2, and C_3 are arbitrary parameters and s_1, s_2, and s_3 are the roots of the characteristic equation

$$\frac{IT}{v(a + \epsilon)} s^3 + \frac{ID}{a + \epsilon} s^2 + \frac{a}{v} s + 1 = 0 \tag{7}$$

Now, while the mathematical representation of the wheel center can be expressed by a unique formula, it is well to bear in mind that the physical appearances of this motion are entirely different depending upon the positive or negative, real or imaginary values of the characteristic equation. Hence the necessity for discussing the values of s_1, s_2, and s_3 in terms of the coefficients of the characteristic equation, and as a consequence, as function of six parameters T, D, ϵ, a, I, v which characterize the tire, the wheel and the airplane speed.

It should be noted that the characteristic equation has no positive coefficients, hence can have no positive root; so if this equation has real roots, they must naturally be negative. On the other hand, the equation has always at least one real root since its first member varies continuously from $-\infty$ to $+\infty$.

Suppose $s_1 = -\alpha$ is this root.

From the point of view of real roots there can be only two possible cases:

(1) Three negative real roots.

The general solution is then of the form

$$z = C_1 e^{-\alpha t} + C_2 e^{-\beta t} + C_3 e^{-\gamma t}$$

α, β, and γ being positive numbers. Whatever the initial conditions that cause the variations of C_1, C_2, C_3, the motion of the wheel is, in this case, an aperiodic and convergent oscillation.

(2) One negative real root $s_1 = -\alpha$ and two conjugate imaginary roots.

It is known that, when s_2 and s_3 are conjugate imaginary, of the form $\lambda \pm \omega i$, the two terms $C_2 e^{s_2 t}$ and $C_3 e^{s_3 t}$ combine to give a term of the form

$$Be^{\lambda t}\sin(\omega t + \varphi)$$

ω being a number essentially positive and λ a positive or negative real number.

The general solution can then be written as

$$z = Ae^{-\alpha t} + Be^{\lambda t}\sin(\omega t + \varphi)$$

the three constants of integration which depend upon the initial conditions of motion, are the parameters A, B, φ.

From the point of view of the physical appearances of the motion this case is split in two:

(a) If $\lambda < 0$ the oscillations are convergent periodic

(b) If $\lambda > 0$ the oscillations are divergent periodic

From among the possible forms of the solution z, the only case in which there is a divergence of oscillation is therefore $\lambda > 0$.

Now an attempt will be made to calculate the angular frequency ω and the divergence λ of the oscillations in terms of the coefficients of the equation of motion of the wheel, that is in terms of the characteristic coefficients of the tire and the wheel. The condition of divergence of the oscillations, that is the condition of instability is derived by writing $\lambda > 0$. In all other cases there is convergence, that is stability.

<u>Calculation of divergence λ and frequency ω</u>

As λ and ω are independent of A, B, φ, it is assumed that $A = 0$, $B = 1$, $\varphi = 0$. Hence

$$z = e^{\lambda t}\sin \omega t$$

$$\frac{dz}{dt} = e^{\lambda t}\left[\lambda \sin \omega t + \omega \cos \omega t\right]$$

$$\frac{d^2z}{dt^2} = e^{\lambda t}\left[(\lambda^2 - \omega^2)\sin \omega t + 2 \lambda\omega \cos \omega t\right]$$

$$\frac{d^3z}{dt^3} = e^{\lambda t}\left[\lambda(\lambda^2 - 3\omega^2)\sin \omega t + \omega(3\lambda^2 - \omega^2)\cos \omega t\right]$$

which, written into the equation of motion of the wheel gives an equation of the form

$$e^{\lambda t}\left[g(\lambda,\omega)\sin \omega t + h(\lambda,\omega)\cos \omega t\right] = 0$$

This equation, which must be checked whatever t may be, gives the system

$$\left|\begin{array}{l} g(\lambda,\omega) = 0 \\ h(\lambda,\omega) = 0 \end{array}\right.$$

of two equations that define λ and ω. Hence the system

$$\left|\begin{array}{l} \dfrac{IT}{v(a + \epsilon)}\lambda(\lambda^2 - 3\omega^2) + \dfrac{ID}{a + \epsilon}(\lambda^2 - \omega^2) + \dfrac{a}{v}\lambda + 1 = 0 \\ \\ \dfrac{IT}{v(a + \epsilon)}(3\lambda^2 - \omega^2) + \dfrac{2ID}{a + \epsilon}\lambda + \dfrac{a}{v} = 0 \end{array}\right.$$

which can also be written

$$\left|\begin{array}{ll} 2T\lambda(\lambda^2 + \omega^2) + vD(\lambda^2 + \omega^2) - v\dfrac{(a + \epsilon)}{I} = 0 & (8) \\ \\ T(3\lambda^2 - \omega^2) + 2vD\lambda + \dfrac{a(a + \epsilon)}{I} = 0 & (9) \end{array}\right.$$

The second of these equations gives

$$\omega^2 = 3\lambda^2 + \frac{2vD\lambda}{T} + \frac{a(a + \epsilon)}{IT} \tag{10}$$

which, entered in equation (8), and ω eliminated, gives the equation

$$8T^2I\lambda^3 + 8DTIv\lambda^2 + 2\left[D^2Iv^2 + a(a + \epsilon)T\right]\lambda + (aD - T)(a + \epsilon)v = 0 \tag{11}$$

and defines the values of λ in terms of the velocity v and the characteristic coefficients of the tire and the wheel. With regard to the abscissa v and ordinate λ, this equation represents a cubic passing through the origin, symmetrical with respect to the origin and asymptotic to the axes of the abscisses. It is easy to plot this cubic and to identify the characteristic elements.

Plotting of curve $\lambda(v)$.- The preceding equation is of the second degree in v and can be written

$$2D^2I\lambda v^2 + \left[8DTI\lambda^2 + (aD - T)(a + \epsilon)\right] v + 8T^2I\lambda^3 + 2a(a + \epsilon)T\lambda = 0 \quad (12)$$

Hence the curve of the values of v is easily plotted against λ. It is readily apparent that for each given real value of λ there always are two real values of v, the only condition being that the discriminant be positive, that is

$$16DT^2I\lambda^2 < (a + \epsilon)(aD - T)^2$$

or

$$|\lambda| < \frac{|aD - T|}{4T}\sqrt{\frac{a + \epsilon}{DI}} = |\lambda_m|$$

Maximum or minimum λ.- When

$$\lambda^2 = \frac{(aD - T)^2}{16T^2}\,\frac{a + \epsilon}{DI}$$

the curve $\lambda(v)$ passes therefore through a maximum or minimum M and the value v_m of the corresponding speed is such that

$${v_m}^2 = \frac{8T^2I\lambda^2 + 2a(a + \epsilon)T}{2D^2I}$$

we get, by replacing λ by its value

$$|v_m| = \frac{aD + T}{2D}\sqrt{\frac{a + \epsilon}{DI}} \quad (12a)$$

Tangent at the origin.- This is the straight line of the equation

$$2aT\lambda + (aD - T)v = 0$$

hence its slope is

$$\frac{T - aD}{2aT}$$

Representative curve.- In consequence of this value of the slope at the origin, the representative curve can assume different shapes, depending upon whether $aD > T$ or $aD < T$.

First case: $aD > T$

The curve $\lambda(v)$ has the shape represented in figure 9. The velocity v being necessarily positive, it is seen that λ is always negative.

Second case: $aD < T$

The curve $\lambda(v)$ has the form represented by figure 10.

The speed v being necessarily positive, λ is, in this case, always positive. The only condition for divergence of oscillations of the wheel is, as was shown before, that $\lambda > 0$, and the only condition of convergence is $\lambda < 0$. In consequence the condition of divergence or instability is reduced to

$$aD < T$$

and that of convergence or stability to

$$aD > T$$

This result could have been reached more rapidly by application of Routh's general rules (or the equivalent method by Hurwitz) of which the particular application to an equation of the third order with positive coefficients of the form

$$A\frac{d^3z}{dt^3} + B\frac{d^2z}{dt^2} + C\frac{dz}{dt} + Dz = 0$$

reduces to the following rule: the necessary and sufficient condition of convergence is $BC > AD$.

In fact, in the case of equation (6) it is seen that this rule yields immediately $aD > T$ as the condition of convergence. But this method involves the use of the original works by Routh ("On the Stability of a Given State of Motion," Adams Prize Essay, 1877) and his treatise ("Advanced Rigid Dynamics") or the works of Hurwitz.

Moreover, it supplies no accurate data about the variations of λ in terms of the airplane speed v.

Resuming the preceding discussion it is seen that when $aD < T$, the oscillations are unstable, that is the wheel, while rolling, assumes a motion of divergent oscillations.

In practice the aforementioned conclusions are not vigorously checked except for the start of the phenomenon. This is due to the fact that our equations are applicable only to small deformations and cease to be rigorous when the oscillations attain sizeable amplitude. So, in the initial phase there is a phenomenon of unstable oscillations, exhibited by a divergence which can become very accentuated when the velocity is near that of the previously defined speed v_m. But the motion is never infinitely divergent. As soon as the oscillations have reached sufficient amplitude, a state of continuous oscillatory motion is established.

The process was frequently believed to be a phenomenon of resonance, arising from a lack of symmetry of the tire or the wheel or similar causes, and interpreted as such in most theories of front wheel shimmy in automobiles. This explanation cannot be sustained in the face of the severe cases of shimmy over the smoothest of grounds with perfectly balanced wheels. Furthermore, the violence of the phenomena in some of its manifestations and the extent of the speed range in which it can manifest itself, are enough to prove that no resonance phenomenon is involved but rather a phenomenon of vibratory instability. The present theory shows that shimmy can be explained in the most elementary way by the inherent mechanical properties of the tire.

The simplicity of the condition of convergence $aD > T$ is surprising. In particular, it is extremely unusual to find that neither the wheel inertia I, nor the offset ϵ or the velocity v figure in the condition of convergence. This condition gives a rule of extreme simplicity; it is sufficient to follow this rule in the construction of the undercarriage $a > \frac{T}{D}$ in order to eliminate elementary shimmy.

Third intermediary case: $aD = T$

In this case, equation (11) gives immediately $\lambda = 0$.

On the other, equation (10) gives

$$\omega^2 = \frac{a(a + \epsilon)}{IT} = \frac{a + \epsilon}{DI}$$

Therefore the general solution takes the form

$$z = Ae^{-\alpha t} + B \sin(\omega t + \varphi)$$

The term $Ae^{-\alpha t}$ rapidly approaches zero, leaving only the second term which represents a sinusoidal motion whose frequency $\omega = \sqrt{\frac{a + \epsilon}{DI}}$ is not dependent on the airplane speed v.

Frequency ω

The quantity λ can be eliminated in the same manner as ω from equations (8) and (9). Multiplying equation (8) by -3 and equation (9) by 2λ and then adding up, gives the equation

$$vID\lambda^2 + 2\left[a(a + \epsilon) - 4IT\omega^2\right]\lambda + 3v\left[(a + \epsilon) - ID\omega^2\right] = 0$$

On the other hand, equation (9) can also be written

$$3IT\lambda^2 + 2vID\lambda + \left[a(a + \epsilon) - IT\omega^2\right] = 0$$

thus giving two equations of the second degree in λ. It is known that the result of the elimination of x from the two equations

$$ax^2 + bx + c = 0$$

$$a'x^2 + b'x + c' = 0$$

is the relation

$$(ac' - ca')^2 - (ab' - ba')(bc' - cb') = 0$$

obtained by equating the result of these two equations to zero. Applying this formula to the two preceding equations in λ, gives the equation

$$v^2 I \left\{ D \left[a(a + \epsilon) - IT\omega^2 \right] - 9T \left[(a + \epsilon) - ID\omega^2 \right] \right\}^2 = 4 \left\{ v^2 ID^2 - 3T \left[a(a + \epsilon) - 4IT\omega^2 \right] \right\} \left\{ \left[a(a + \epsilon) - 4IT\omega^2 \right] \left[a(a + \epsilon) - IT\omega^2 \right] - 3v^2 ID \left[(a + \epsilon) - ID\omega^2 \right] \right\} \tag{13}$$

which defines the values of ω in terms of v and the characteristic coefficients of the tire and wheel. This equation of the sixth degree in v and ω is reduced to one of the third degree by a change of the coordinates $v^2 = x$ and $\omega^2 = y$. Unfortunately the direct study of this equation proved to be rather difficult, and was therefore abandoned in favor of a more simple method of defining the variations of ω in terms of v. In fact it can be stated that the writer has studied the variations of λ in terms of v, and was thus able to study the variations of ω in terms of λ, and the problem of the variations of the frequency ω with respect to v can be solved the same way.

The velocity v is easily eliminated from equations (8) and (9); hence the equation

$$\left(\lambda^2 + \omega^2\right)^2 + \frac{a + \epsilon}{I}\left(\frac{3}{D} - \frac{a}{T}\right)\lambda^2 - \frac{a + \epsilon}{I}\left(\frac{1}{D} + \frac{a}{T}\right)\omega^2 + \frac{a(a + \epsilon)^2}{I^2 DT} = 0 \tag{14}$$

which defines the values of ω in terms of λ and the characteristic coefficients of the tire and wheel. This equation of the fourth degree in λ and ω is reduced to one of the second degree by changing the coordinates $\lambda^2 = X$ and $\omega^2 = Y$. The result is

$$(X + Y)^2 + \frac{a + \epsilon}{I}\left(\frac{3}{D} - \frac{a}{T}\right)X - \frac{a + \epsilon}{I}\left(\frac{1}{D} + \frac{a}{T}\right)Y + \frac{a(a + \epsilon)^2}{I^2 DT} = 0 \tag{15}$$

The equation of the group of asymptotic directions

$$(X + Y)^2 = 0$$

indicates that the conic involved is a parabola whose axis is parallel to the second bisectrix of the coordinates: $Y = -X$. Only the portion corresponding to the conditions $X > 0$, $Y > 0$, that is located in the first quadrant, will be considered.

This arc of the parabola is easily plotted. For $X = 0$, that is $\lambda = 0$, the two values

$$Y_A = \omega_A^2 = \frac{a(a + \epsilon)}{IT} \qquad \text{represented by point A}$$

$$Y_B = \omega_B^2 = \frac{a + \epsilon}{ID} \qquad \text{represented by point B}$$

are obtained.

By referring to equation (13) or simply to equation (10) or equations (8) and (9) it is easy to identify that the first of these values corresponds to zero velocities and the second to infinite velocities.

On the other hand,

$$Y_B - Y_A = \frac{a + \epsilon}{IDT}(T - aD)$$

hence, when $aD < T$, that is in the case of divergence of the oscillations, point B is above point A (fig. 11); while, when $T < aD$, that is convergence, point B is below point A (fig. 13).

Arranging equation (15) according to the decreasing powers of Y under the form $AY^2 + BY + C = 0$ and posting the condition $B^2 - 4AC > 0$ to be fulfilled so that the values of Y are real, results in

$$X_m = \lambda_m^2 \leqslant \frac{(aD - T)^2}{16T^2} \frac{a + \epsilon}{DI}$$

a value already obtained as maximum of λ^2 in the study of λ in terms of v.

The corresponding value of $Y_m = \omega_m^2$ is readily obtained by considering equation (15) developed under the form

$$AY^2 + BY + C = 0, \quad \text{hence} \quad Y_m = - \frac{B}{2A}$$

which gives

$$Y_m = \frac{a + \epsilon}{2I}\left(\frac{1}{D} + \frac{a}{T}\right) - X_m$$

hence

$$Y_m = \omega_m{}^2 = \frac{a + \epsilon}{16DIT^2}\left(7T^2 + 10aDT - a^2D^2\right)$$

Returning to the study of variations of Y, that is of ω^2 when v varies from zero to infinity, it is apparent that the representative point of the variations of ω^2 and λ^2 (figs. 11 and 13) passes from A to B along the parabolic arc defined by equation (15) and that B is above A when $aD < T$ and below in the opposite case. In order to define the shape of the representative curve of the variations of the frequency ω in terms of velocity v, it is necessary to establish whether the variations of ω^2 are increasing or decreasing as in figures 11 and 13 or can pass through a minimum L as in figure 15. As to the possibility of a maximum it is apparent that, geometrically, it is excluded by reason of the general position of the parabola.

For it to have a minimum L it is necessary and sufficient that the slope of the tangents to A and B have the same sign. So, when equation (15) is differentiated, the value of the slope p at any one point is

$$p = \frac{dy}{dx} = \frac{\frac{a + \epsilon}{I}\left(\frac{3}{D} - \frac{a}{T}\right) + 2(X + Y)}{\frac{a + \epsilon}{I}\left(\frac{1}{D} + \frac{a}{T}\right) - 2(X + Y)}$$

Since the coordinates X and Y of points A and B are known, the preceding formula gives the slopes p_A and p_B; forming the product gives

$$p_A \times p_B = \frac{(aD + 3T)(aD - 5T)}{(aD - T)^2}$$

which for $p_A p_B > 0$ reduces to $aD > 5T$.

This condition involves a fortiori $aD > T$; hence there can be no minimum except in the case of convergence.

The discussion will be concluded with an examination of the possibility of aperiodic solutions. For this possibility to exist it is necessary and sufficient that $Y = \omega^2$ can assume negative values, that is that the parabolic arc AB intersect axis OX. This geometrical condition is expressed by the double analytical condition that the equation in X obtained by making $Y = 0$ in equation (15) can have at least one real and positive root.

The two roots of this equation are both real or imaginary; to be real it is necessary and sufficient that the discriminant be positive, which gives the condition

$$9T^2 - 10aDT + a^2D^2 > 0$$

This condition requires that

$$aD < T$$

or

$$aD > 9T$$

As, on the other hand, the product of the roots $\frac{a(a + \epsilon)^2}{I^2DT}$ is positive, the roots are either both positive or negative. To be positive, their sum must be positive, which gives the condition $\frac{3}{D} - \frac{a}{T} < 0$ that is $aD > 3T$.

To satisfy the required double analytical condition it is therefore sufficient to know the unique relation $aD > 9T$; this is the condition necessary and sufficient for the existence of aperiodic solutions for suitably chosen v. These velocities of aperiodic conditions are, obviously, those comprised between the velocities v_1, and v_2, positive solutions of the biquadratic equation obtained by making $\omega = 0$ in equation (13). The corresponding convergences are obtained the same way by making $\omega = 0$ in equation (14).

Now the curves $\omega(v)$ representative of the variations of the frequency ω or, which is the same, the curves ω^2 in terms of velocity v can be accurately plotted. It is accomplished by associating to each curve $\omega^2(v)$ the corresponding curve $\omega^2(\lambda^2)$.

The foregoing discussion distinguished four cases:

(1) $aD < T$ divergent periodic oscillation (figs. 11 and 12)

(2) $T < aD < 5T$ convergent periodic oscillation with decreasing frequency (figs. 13 and 14)

(3) $5T < aD < 9T$ convergent periodic oscillation with maximum frequency (figs. 15 and 16)

(4) $9T < aD$ convergent aperiodic oscillation when $v_1 \leqslant v \leqslant v_2$ convergent periodic in the other cases. (figs. 17 and 18)

B. Complete Theory of Elementary Shimmy

The equations (1), (2), and (3) which form the basis of the simplified elementary shimmy were obtained by assuming the moment torque $\underline{M}$ arising from the reactions of torsion and turning, a torque exerted directly on the tread by the ground to be negligible. To take this moment $\underline{M}$ into account, equations (1) and (3) must be suitably modified, while equation (2) remains the same.

First of all, the plus sign is placed before M when a restoring torque due to a positive displacement z is involved, just like the force F was given the plus sign when a restoring torque due to a positive displacement z was involved.

Next, consider equation (3); its first term represents the angular acceleration of the wheel about the caster axis. The quantity $-F(a + \epsilon)$ was the generating moment of this acceleration. In the complete theory, the moment -M simply added to this moment in such a way that this equation becomes

$$\frac{I}{a}\frac{d^2z}{dt^2} = -F(a + \epsilon) - \underline{M}$$

As to equation (1), it expressed the relation between the direction dy/dx of the contact surface and force F. It had been assumed by writing this equation that the drift, that is the orientation of the contact surface immediately followed the variations of force F; it implicitly implied that the turn was instantaneous, or in other words, implicitly attributed an infinite value to the turn coefficient R. To understand the line of reasoning that leads to the equation desired, we shall start with the analysis of a particularly simple phenomenon: drift.

Complete Study of Drift

Consider a pneumatic tire at rest (fig. 19). A force F normal to its plane is applied and it is made to roll while maintaining its plane parallel to a fixed direction OP. Up to now it had been assumed that the contact surface shifts along a straight line OM forming an angle of drift $\delta = DF$ with the plane of the wheel, that is instantaneous drift was assumed. But experience indicates that the trajectory of the contact surface or rather its center is actually a curve CAB tangential to OP in O and asymptotic to O'M' parallel to OM. This contact area can turn only according to the law of turning

$$\beta = d\,\frac{dy}{dx} = R\underline{M}\,dx$$

that is

$$\frac{d^2y}{dx^2} = R\underline{M}$$

But this couple $\underline{M}$, the generator of turning, is itself the difference of two effects: the accompanying couple $F\epsilon$ and the torsional couple of the tread equal to

$$\frac{1}{S}\frac{dy}{dx}$$

in such a way that

$$\underline{M} = F\epsilon - \frac{1}{S}\frac{dy}{dx}$$

hence

$$\frac{d^2y}{dx^2} = R\left[F\epsilon - \frac{1}{S}\frac{dy}{dx}\right]$$

or

$$S\,\frac{d^2y}{dx^2} + R\,\frac{dy}{dx} = RS\epsilon F$$

This then is the differential equation of the trajectory OAB of the center of the contact surface in drift. This equation is easily integrated

$$y = S\epsilon Fx + \frac{S^2\epsilon F}{R}\left(e^{-\frac{R}{S}x} - 1\right)$$

an equation that closely represents a curve having the form of that plotted in figure 19. Moreover, this equation makes it possible to present an unusually interesting result: consider the drift dy/dx; when x increases indefinitely, dy/dx tends toward $S\epsilon F$. Comparison with the formula

$$\frac{dy}{dx} = DF$$

it is seen that

$$S\epsilon = D$$

This relation is retained and applied repeatedly in the subsequent study. Another significant feature of this relation is that it reduces the number of characteristic parameters of the ordinary tire by one. Take for example, T, D, ϵ, R or else T, D, S, R. The equation of drift then reads

$$y = DFx + \frac{DS}{R} F\left(e^{-\frac{R}{S}x} - 1\right)$$

In the light of this digression on drift we now return to the most general motion problem. The path curvature of the center of the contact surface is always

$$\frac{d^2y}{dx^2} = -R\underline{M}$$

The generating moment $\underline{M}$ of turning of the contact surface being still the difference of two effects, the torsion couple of the tread equal to α/S, α being the angle between the axis of contact surface and wheel plane, that is $\alpha = \frac{dy}{dx} + \frac{z}{a}$, and the accompanying couple F.

Hence

$$\underline{M} = \frac{1}{S}\frac{dy}{dx} + \frac{z}{Sa} - F\epsilon$$

But $\frac{d^2y}{dx^2}$ can still be written $\frac{1}{v^2}\frac{d^2y}{dx^2}$ if v is constant. Hence the system of four equations

$$\frac{1}{v^2}\frac{d^2y}{dt^2} = -RM \tag{16}$$

$$z - y = TF \tag{17}$$

$$\frac{I}{a}\frac{d^2z}{dt^2} = -F(a + \epsilon) - \underline{M} \tag{18}$$

$$\underline{M} = \frac{1}{Sv}\frac{dy}{dt} + \frac{z}{Sa} - F\epsilon \tag{19}$$

Entering the values of F amd M in equations (17) and (19) and then in equations (16) and (18) gives

$$\left|\begin{aligned} &\frac{1}{Rv^2}\frac{d^2y}{dt^2} + \frac{1}{Sv}\frac{dy}{dt} + \frac{\epsilon}{T}y + \left(\frac{1}{Sa} - \frac{\epsilon}{T}\right)z = 0 \qquad (20) \\ &\frac{1}{Sv}\frac{dy}{dt} - \frac{a}{T}y + \frac{I}{a}\frac{d^2z}{dt^2} + \left(\frac{1}{Sa} + \frac{a}{T}\right)z = 0 \qquad (21) \end{aligned}\right.$$

The elimination of y from the two preceding equations then leaves the differential equation in z defining the motion of the wheel. Thus

$$\frac{I}{Rav^2}\frac{d^4z}{dt^4} + \frac{I}{Sav}\frac{d^3z}{dt^3} + \left(\frac{I\epsilon}{Ta} + \frac{1}{SRav^2} + \frac{a}{TRv^2}\right)\frac{d^2z}{dt^2} + \frac{a + \epsilon}{TSv}\frac{dz}{dt} + \frac{a + \epsilon}{TSa}z = 0 \tag{22}$$

Note 1.- Setting $R = \infty$ in the preceding equation and bearing in mind the relation $S\epsilon = D$, yields the equation (6) of the simplified theory.

Note 2.- The elimination of y from equations (20) and (21) requires algebraic equations which can be simplified by the following considerations; the elimination must result in a linear differential equation in z. If n is the degree of this equation it will have n particular solutions of the form $z = Ae^{\alpha t}$, the coefficients α being real or imaginary. Conversely, the elimination of z leaves a unique equation in y. Equation (20) indicates that, if y is of the form $y = A_1e^{\alpha t}$, z is also of the form $z = Ae^{\alpha t}$. Equation (21) indicates that, when z is of the form $z = Ae^{\alpha t}$, there also is a solution y of the form $A_1e^{\alpha t}$. Therefore, every solution of the differential equation in z is also a solution of the differential equation in y and conversely. Thus these two equations are deduced by simply changing y for z and vice versa. The elimination of z is much easier; simply enter its value in equation (20) and then substitute in equation (21). This method makes it possible to write equation (22) very quickly.

Stability of oscillation.- The preceding equation is a linear differential equation of the fourth order of the form

$$a_0z^{IV} + a_1z^{III} + a_2z^{II} + a_3z^{I} + a_4z = 0$$

and all its coefficients are positive.

Referring to Routh's "Advanced Rigid Dynamics" it is seen that the conditions necessary for such an oscillation to be stable are

$$a_1a_2 > a_0a_3$$

$$a_1a_2a_3 > a_0a_3^2 + a_1^2a_4$$

Applied to equation (22) the first of these conditions gives, after reductions,

$$\frac{I\epsilon}{a} + \frac{T}{SRav^2} > \frac{\epsilon}{Rv^2}$$

which, with allowance for the fundamental relation $S\epsilon = D$, can be written

$$v^2 > \frac{aD - T}{IRD} \tag{23}$$

The second of these conditions yields, after reductions

$$\frac{I\epsilon}{Ta} + \frac{1}{SRav^2} > \frac{\epsilon}{TRv^2} + \frac{I}{Sa^2}$$

or

$$(IRv^2 - a)(S\epsilon a - T) > 0$$

that is with the fundamental relation previously advanced,

$$(IRv^2 - a)(aD - T) > 0 \tag{23a}$$

IV. COMPARISON OF THE PRESENT THEORY WITH THE EXPERIMENTAL RESULTS

Let us interrupt the examination of these two stability conditions, and begin by studying the second. To interpret it, assume that speed u is defined by the equality

$$u^2 = \frac{a}{IR}$$

so the discussion can be summed up as follows:

First case: $aD > T$

The condition (23a) is confirmed if $v > u$.

Second case: $aD < T$

The condition (23a) is confirmed if $v < u$.

It is readily apparent that in both cases equation (23) is confirmed when the condition (23a) is confirmed. Therefore the conditions of the stability of oscillation are as follows:

First case: $aD > T$ with $v > u$

Second case: $aD < T$ with $v < u$

This u is termed the inversion speed of stability. The determination of the order of magnitude of this inversion speed u for the usual pneumatic tires seems to be essential. But unfortunately no exact determination of the value R is available so that this value could be introduced in the formula

$$u^2 = \frac{a}{IR}$$

Nevertheless, it shall be shown that the apparently very different experimental results can be used to define the order of magnitude of the inversion speed for the ordinary pneumatic tires.

Referring to the earlier study of drift and to figure 19, and assuming that the equation for the curve OAB is given, it is readily seen that $\frac{CD'}{DD'} = e^{-\frac{R}{s}x}$ at any point C of this curve. Now, for the usual pneumatic tires the curve OAB rapidly tends toward its asymptotic O'D'M', and according to the data given by various experimenters it seems permissible to assume that after a travel of 50 cm

$$\frac{CD'}{DD'} < \frac{1}{4}$$

which, with the centimeter as the unit of length, gives

$$e^{\frac{R}{S}50} > 4$$

hence

$$50\,\frac{R}{S} > 1.40, \quad \text{that is} \quad R > 0.028S$$

So, if the values of S for the usual tires are known, the problem can be solved. Unfortunately, no such determinations are as yet available. But it was seen that $S\epsilon = D$. Thus the preceding relation becomes

$$R > 0.028\,\frac{D}{\epsilon}$$

On the other hand, it has been indicated that if l denotes the length of surface contact

$$\frac{l}{6} < \epsilon < \frac{l}{2}$$

So a length of 3 cm for ϵ is an altogether admissible value and probably very close to the real value in a great number of cases. The preceding inequality consequently becomes

$$R > \frac{0.028}{3} D \quad \text{or} \quad R > 0.0093D$$

Experiments made on automobile tires have shown that under a side force of 100 kg the drift $\delta = DF$ reaches a value approaching 3°, that is

$$\frac{3\pi}{180} = 0.0524 \text{ radians}$$

Using the C.G.S. system of units the coefficient D is

$$D = \frac{0.0524 \text{ radians}}{100.000 \times 981 \text{ dynes}} = 5.35\ 10^{-10} \quad \text{(C.G.S. system)}$$

hence

$$R > 5.10^{-12} \quad \text{(C.G.S. system)}$$

Supposing now that the caster length a is 5 cm and the inertia I is 10 kg at 1 m from the axis, that is 10^8 (C.G.S. system). In that case

$$u^2 = \frac{a}{IR} = \frac{5}{10^8 \times 5 \times 10^{-12}} = 10^4$$

$u = 100$ cm/sec $= 1$ m/sec or 3.6 km/hr

With a 10 cm caster length it would be

$$u = 1.4 \text{ m/sec} = 5.1 \text{ km/hr}$$

With a 20 cm caster length

$$u = 2 \text{ m/sec} = 7.2 \text{ km/hr}$$

The essential part to remember is that the inversion speed u is small. This is, as shall be shown, an extremely important result and it can be regarded as incontestable, although R has been derived by indirect means. The form

$$u^2 = \frac{a}{IR}$$

of the equation used to define u has, in fact, shown that a given variation of R involves only a very small variation of u, so that, for R one-fourth as large, for example, the real value of u would only be doubled. Furthermore, the results of this analysis, and in particular the 1 m/sec obtained for the inversion, coincide exactly with the experimental curves published by John Wylie of the Douglas Aircraft Company (ref. 1). That writer gives the exact coefficient of friction K of the hydraulic damper necessary to avoid shimmy on the landing gear of the OA-4A airplane. This curve is reproduced in figure 20. There is no need for a damper at velocities below 1 m/sec., but there is for all speeds above this value. The case exhibited by this airplane was the case $a < \frac{I}{D}$. Therefore Wylie's curve evidences an inversion speed with stability of oscillations below this speed and instability above it. Moreover, the inversion speed resulting from Wylie's curve is exactly coincident with that evaluated in the present numerical example.

The same writer also gives a similar curve for the nose wheel of the Douglas DC-4. The inversion speed of the stability for this airplane is around 1.80 m/sec, hence is still close to that evaluated in the present article.

The Douglas DC-4 in 1940 was the largest airplane which had been equipped with a nose wheel landing gear. It was a four-engine all metal, low wing monoplane. Each engine developed 1165 hp., its weight was 29.6 tons and its one nose wheel weighed 300 kg. It was designed for 47 passengers, had a 382 km/hr top speed and 119 km/hr landing speed.

A similar curve was obtained by E. Maier and M. Renz in Germany during the war, on the nose wheel of the Douglas DB-7 "Boston." The inversion speed seems to have been near 4 m/sec.

As a result of these experimental data the existence of a low inversion speed u can be regarded as proved by experiment. Therefore referring to the stability condition (23) it can be seen that, in order to eliminate shimmy at all speeds above u, it is sufficient to realize the condition $aD > T$. At speeds below u, which are low speeds, it is evident that shimmy should not be very annoying and can, perhaps pass completely unnoticed in certain cases, because, as will be shown later, simple relatively moderate friction in the hinge is enough to damp it out instantly. The consequence of this analysis, that is, the condition

$$aD > T$$

can be practically regarded as the fundamental stability condition of the wheel.

Can this condition be easily achieved for the permissible values of caster lengths a? This is the next problem to be treated.

If the exact values of the coefficients T and D for the usual tires are known, the answer is immediate, but unfortunately there are no such data. So far it has been possible to evaluate the order of magnitude of the coefficients D, S, R, and ϵ on the basis of earlier experiments, but still we know nothing of coefficient T. However, the difficulty can be overcome by suitable interpretation of seemingly very dissimilar experiments.

We refer to the report by B. v. Schlippe and R. Dietrich, entitled The Mechanics of Pneumatic Tires (ref. 2) which is a detailed study of the behavior of the pneumatic tire. Unfortunately, the authors failed to give the simple properties that control the mechanics of the tire, as they did not take up the study of the most complex phenomenon, shimmy. Nevertheless, their report supplies some significant information because it contains a certain amount of numerical data. They made experiments and measurements on a 260 × 85 tail wheel fitted with a Continental balloon tire with small longitudinal grooves carved around the circumference. The tire pressure was 2.5 atm and the load 180 kg. The wheel was connected to different dynamometers and kept stationary, and rolling was accomplished by a rotating drum 90 cm in diameter, covered with emery paper to assure good adhesion.

The first experiment by these writers to which attention is called, is the following: when a tire is made to roll by keeping the plane of the wheel fixed and imposing on it a rectilinear trajectory making a certain angle θ with the wheel plane (fig. 22), the center M of the contact surface describes a curvilinear path MM' having as asymptote the straight line (D) located a distance m from the trajectory OO' of the wheel center.

The authors established, by experiment, the proportionality

$$m = K\theta$$

K being equal to 10 if θ is expressed in radians and m in cm.

When considering the side force F necessary to keep the wheel on the trajectory OO', the angle θ is, obviously, the angle of drift, hence $\theta = DF$. On the other hand, m is evidently the displacement of the lateral elasticity and consequently $m = TF$, hence

$$\frac{m}{\theta} = \frac{T}{D}$$

hence

$$m = \frac{T}{D}\,\theta$$

Their coefficient K is none other than the quotient T/D, and the experiment in figure 22 gives this quotient directly. The quotient for this wheel is, thus, equal to 10 cm, and the caster length necessary to eliminate shimmy is $a \geqslant 10$ cm.

This particular example indicates that the caster lengths to which the present theory leads are of a reasonably approximate magnitude, at least in certain cases. Hence nose-wheel shimmy can be probably eliminated by a simple modification of the position of the pivoting axis of the wheel, thus making the use of shimmy dampers unnecessary.

The experiments by Schlippe and Dietrich are of further interest for another reason: they enable the characteristic coefficients T, D, S, R, and ϵ of the tire to be found indirectly, which, in the absence of more adequate measurement, helps in defining the approximate magnitude of these fundamental factors. Their direct measurement of the lateral elasticity of the wheel indicated that a side force F of 2×32.5 kg, or 65 kg, was necessary to produce a 1 cm lateral displacement Δ of the tread with respect to the wheel. Now if $\Delta = TF$, we get in C.G.S. units

$$T = \frac{1}{65000 \times 981}$$

or

$$T = 157 \times 10^{-10} \quad \text{(C.G.S. system)}$$

Since $\frac{T}{D} = 10$, we get $D = 15.7 \times 10^{-10}$ (C.G.S. system)

These authors also indicated that, to maintain an angle $\theta = 1^{\circ}$ or 0.0175 radians in the experiment described above (fig. 22), a side force of 11.4 kg and a torque having a moment C of 49.7 cm × kg must be exerted. So, since $\theta = DF$, we should have

$$0.0175 = D \times 11.4 \times 981.000$$

hence

$$D = 15.6 \times 10^{-10} \quad \text{(C.G.S. system)}$$

The agreement between this figure and that obtained independently some lines back is perfect.

The accompanying torque measured by these authors was $\underline{C} = 49.7$ cm. kg. It is known that

$$\underline{C} = \epsilon F$$

hence $\epsilon = \dfrac{49.7}{11.4}$ or $\epsilon = 4.35$ cm.

This figure is compared with the length l of the tread or ground contact surface. The authors indicate that this length was about 9 cm. Hence

$$\frac{l}{\epsilon} = \frac{9}{4.35} = 2.1$$

This result is in close agreement with the formula

$$2 < \frac{l}{\epsilon} < 6$$

given at the beginning of the present article while analyzing the accompanying torque.

The torsional elasticity was measured by the difference in lateral displacement Z of the lead point and Z' of the tail point of the tread contact in such a way that if l is the length of the tread, the torsion angle α is such that

$$\sin \alpha = \frac{Z - Z'}{l}$$

therefore, for the small angles

$$\alpha = \frac{Z - Z'}{l}$$

For these conditions the authors set up the formula

$$\underline{M} = d(Z - Z')$$

where $\underline{M}$ is the moment necessary to produce the torsion and d is a constant equal to 317 kg, where Z and Z' are expressed in cm and $\underline{M}$ in cm. kg, that is $d = 317 \times 981000$ (C.G.S.). Now $Z - Z' = l\alpha$, hence the formula giving $\underline{M}$ becomes

$$\underline{M} = dl\alpha$$

However, since $\alpha = S\underline{M}$ and $l = 9$ cm,

$$S = \frac{1}{dl} = \frac{1}{9 \times 317 \times 981000} \quad \text{that is,} \quad S = 3.57 \times 10^{-10} \quad \text{(C.G.S. system)}$$

On the other hand, $D = S\epsilon$ and $\epsilon = 4.35$ cm. Therefore,

$$D = 4.35 \times 3.57 \times 10^{-10} \quad \text{that is} \quad D = 15.5 \times 10^{-10} \quad \text{(C.G.S. system)}$$

Two independent methods have already given

$$D = 15.7 \times 10^{-10} \quad \text{and} \quad D = 15.6 \times 10^{-10}$$

The agreement between these three figures is really remarkable, it even seems to exceed the probable accuracy of the measurement. At any rate, these results give a completely satisfactory experimental check of the present theory.

The one characteristic left to define is the turn coefficient R. Its determination will be based upon the following experiment: by causing the tire to roll against or on a circular wheel of 1 m radius, the authors claimed that the motion produced an axial force F of 21.2 kg, tending to keep the tire away from center of the periphery.

If, as seems likely, according to the conditions of the experiment, it is assumed that the generating moment of $\underline{M}$ of the rotation arises solely from the accompanying torque

$$F\epsilon = 21.2 \times 981000 \times 4.35 = 90.5 \times 10^6 \quad \text{(C.G.S. system)}$$

the coefficient R can be deduced by the formula

$$\frac{d^2y}{dx^2} = R\underline{M}$$

If ρ signifies the curvature radius of the track the formula $\frac{d^2y}{dx^2} = \frac{1}{\rho}$ must be used, hence

$$R = \frac{1}{\rho \underline{M}} \quad \text{that is} \quad R = \frac{1}{100 \times 90.5 \times 10^6} \quad \text{or} \quad R = 1.1 \times 10^{-10} \text{ (C.G.S. system)}$$

It will be noted that this value of R is in good agreement with the inequality $R > 5 \times 10^{-12}$ indicated previously. Still, this determination does not offer the same degree of certainty as the preceding ones: in fact since it was necessary to assume that the wheel was exactly perpendicular to the radius of the track, any error in this special circumstance will modify the centrifugal force F by the superposition of a drift effect. To eliminate this potential source of error completely it is necessary to assure the equality of the centrifugal force F by changing the direction of rotation. In the case of a minor discrepancy, the average should be taken.

U.S. Reports on Shimmy

The problem of nose-wheel shimmy has already formed the object of numerous theoretical and experimental studies, expecially in the United States where the so-called tricycle landing gear was born. It was studied in great detail by the Douglas Aircraft Company, then a little later, by the Lockheed Aircraft Corporation. On the occasion of these studies a mathematical theory of shimmy was suggested by Wylie (ref. 1) and by Arthur Kantrowitz of the Langley Memorial Aeronautical Laboratory of the National Advisory Committee for Aeronautics (NACA Rep. 686).

Because it is the only theory giving a numerical account of the phenomena of shimmy, it is deemed practical to reproduce a literal exposition of Kantrowitz's article. This theory is proposed as being "based on the discovery of a new phenomenon called kinematic shimmy." Here is the exposition of Kantrowitz.

1. Kinematic Shimmy

Some preliminary experimental results on shimmy were obtained by the N.A.C.A. with the aid of the belt-machine apparatus shown in figure 23. This machine consists of a continuous fabric belt mounted on two rotating drums and driven by a variable-speed electric motor. Provision is made for rolling a castering wheel up to about 6 inches in diameter on the belt in such a way that it is free to move vertically but not horizontally.

On this belt machine, the following phenomenon (see fig. 23) was discovered while pushing the belt very slowly by hand. With the wheel set at an angle with the belt as in (a) and the belt pushed slowly, the bottom of the tire would deflect laterally as is shown in (b). When the belt was pushed farther, the wheel straightened out gradually as is shown in (c). The bottom of the tire would then still be deflected, however, and the wheel would continue to turn as in (d). The wheel would thus finally overshoot, as shown in (e) and (f). The process would then be repeated in the opposite direction.

Figure 24 is a photostatic record of the track left by the bottom of the tire on a piece of smoked metal. Two things will be noticed: First, that the bottom of the tire did not skid; and, second, that the places where the wheel angle is zero (indicated by zeros on the track) correspond roughly to the places where the lateral deflection of the tire is a maximum. Thus the wheel angle lags the tire deflection by one-quarter cycle.

It was noticed that the oscillation could be interrupted at any point in the cycle by interrupting the motion of the belt without appreciably altering the phenomenon. From this observation it was deduced that dynamic forces play no appreciable part in this oscillation.

The distance along the belt required for one cycle was also found not to vary much with caster angle or caster length. (See fig. 25.) Caster length was therefore considered not to be of fundamental importance in this type of oscillation.

It should be pointed out that, in order to observe the kinematic shimmy, lateral restraint of the spindle is necessary to prevent the spindle from moving laterally when the bottom of the tire is deflected and thus neutralizing the tire deflection. This restraint is supplied by the dynamic reaction of the airplane when the airplane is moving forward rapidly but is not ordinarily present when the airplane is moving forward slowly. It has been observed, however, on airplanes towed slowly with two towropes so arranged as to provide some lateral restraint.

Figure 23(a) shows that, when the center line of the wheel is at an angle θ (see fig. 25) with the direction of motion, the bottom of the tire deflects. This situation is represented schematically in figure 26. It is seen that a typical point on the peripheral center line must have a component of motion perpendicular to the wheel center line if the tire is not to skid. Thus

$$d\lambda = -\sin\theta \, ds$$

(The minus sign follows from the conventions used as shown in fig. 25.) Since only small oscillations are to be considered, the approximation

$$\frac{d\lambda}{ds} = -\theta \tag{1}$$

may be substituted.

The effect of tire deflection on θ will now be considered. For the purposes of rough calculation, it will be assumed that, as illustrated in figure 27, the projection of the peripheral center line on the ground is a circular arc intersecting the wheel central plane at the extremities of the projection of the tire diameter. (See fig. 23(d).) Thus, in figure 27, r is the tire radius. (It will be assumed for the time being that the caster length and the caster angle are zero.) Now if the tire is deflected in the form of a circular arc, then the condition that the torque about the spindle axis be zero is that the strain be symmetrical about the projection of the wheel axle on the ground. Clearly, if the wheel is displaced, it will be turned about the spindle axis by the asymmetrical elastic forces until, if it is allowed time to reach equilibrium, the symmetrical strain condition is reached. Thus, if the tire is deflected an amount λ as in figure 27(a) and if the wheel rolls forward a distance ds to the condition shown in figure 27(b), in order for the strain to remain symmetrical the wheel must turn about the spindle axis an amount $d\theta$. From figure 27, $R d\theta = ds$. The value of R may be readily obtained from geometry in terms of r and λ. Thus $R^2 = r^2 + (R - \lambda)^2$, from which, if $\lambda^2 \ll r^2$, it is seen that $R = r^2/2\lambda$. Then substituting for R,

$$\frac{d\theta}{ds} = \frac{2}{r^2}\lambda \tag{2}$$

If the caster length is finite, the strain will not be symmetrical about the axle, as was assumed here, but will be symmetrical with respect to some line parallel to the axle but a certain fixed distance ahead. Hence, the essential elements of the geometry are unchanged and all the reasoning that led to equation (2) is still valid for this case.

Since the phenomena represented by equations (1) and (2) occur simultaneously, they must be combined to get the total effect. Thus

$$\frac{d^2\theta}{ds^2} = -\frac{2}{r^2}\theta \tag{3}$$

This differential equation corresponds to a free simple harmonic oscillation occurring every time the wheel moves a distance $S = \frac{2\pi}{\sqrt{2/r^2}} = \pi r\sqrt{2}.$

Measurements of the space interval S of kinematic shimmy have been made for three similar tires of the type illustrated in figure 23. These tires all had radii of approximately 2 inches so that the theoretical interval was about 0.74 foot. Their experimental intervals were 0.65 foot, 0.74 foot, and 0.79 foot. This agreement is closer than might have been anticipated in view of the roughness of the assumption. It will be seen from equation (1) that λ is one-fourth cycle out of phase with θ, which is in agreement with the information obtained from figure 24.

In order to take account of tires for which the assumption made concerning the projection of the peripheral center line is not quantitatively valid, an empirical constant K will be used in place of $2/r^2$ in equations (2) and (3), thus obtaining

$$\frac{d\theta}{ds} = K\lambda \tag{4}$$

and

$$\frac{d^2\theta}{ds^2} = -K\theta \tag{5}$$

The constant K can be measured by observing the space interval of kinematic shimmy. Where experimental values of K are available, they will be used rather than the rough theoretical value $2/r^2$.

2. Dynamic Shimmy

In the foregoing derivation for the oscillation called kinematic shimmy, it was assumed that the strain of the tire was always symmetrical, that is, the wheel was moving so slowly that any torque arising from dynamic effects involved in the oscillation would be negligible. For this case, from equation (4)

$$\left(\lambda - \frac{1}{K}\frac{d\theta}{ds}\right) = 0$$

If, now, the wheel is assumed to be moving at a velocity such that the effect of the moment of inertia about the spindle axis is significant, then the strain can no longer be symmetrical and, for small asymmetries, the torque exerted by the tire on the spindle will be proportional to the amount of the asymmetry. Thus, the value in parentheses will no longer be zero but it can be assumed that it will be proportional to the dynamic torque; hence

$$\frac{d^2\theta}{dt^2} = C_1\left(\lambda - \frac{1}{K}\frac{d\theta}{ds}\right)$$

where C_1 is an appropriate constant of proportionality and includes the moment of inertia about the spindle axis I. If the forward velocity V of the wheel is constant

$$\frac{d^2\theta}{dt^2} = V^2\frac{d^2\theta}{ds^2}$$

and

$$V^2\frac{d^2\theta}{ds^2} = C_1\left(\lambda - \frac{1}{K}\frac{d\theta}{ds}\right) \qquad (6)$$

The constant C_1 may be determined by deflecting the bottom of the tire a known amount, moving the wheel forward, and balancing the torque $\underline{M}$ exerted by the tire on the spindle so that θ stays constant. The method of deflecting the bottom of the tire a known amount will be described later. In this case (6) becomes

$$\frac{M}{I} = C_1\lambda \tag{6a}$$

from which C_1 may be found. It has been observed that C_1 increases with increasing caster angle. Thus, for a tire like the one in figure 23, C_1 was 71,000 radians per second^2 per foot for a caster length L of 0.17 inch (caster angle, 5°; no fork offset) and was 104,000 radians per second^2 per foot for caster length 0.68 inch (caster angle, 20°).

In the study of kinematic shimmy, it was also seen that the only change in λ was due to the fact that a component of the forward motion was perpendicular to the central plane of the wheel. This circumstance is expressed by equation (1). It is, however, found that, if the spindle is clamped at $\theta = 0^\circ$ and the bottom of the tire is deflected, the deflection will gradually neutralize itself; that is, the bottom of the tire will roll under the wheel. Thus the asymmetrical $\left(\frac{d\theta}{ds} = 0\right)$ strain that exists in this case contributes to $d\lambda/ds$. The case of θ and $d\theta/ds$ zero is illustrated in figure 28. If the effect is again supposed to be proportional to the cause, there is obtained

$$\frac{d\lambda}{ds} = -C_2\lambda \tag{7}$$

In this equation, the constant C_2 is a geometrical constant of the tire that can be obtained from static measurements. The order of magnitude of C_2 can be obtained by assuming that the periphery of the tire intersects the extremity of the extended central plane of the wheel. In that case $C_2 = \frac{1}{r}$.

If θ is not zero, there will be a component of the forward motion contributing to $d\lambda/ds$. As in equation (1), this component will be $-\theta$. Adding this component to the part of $d\lambda/ds$ due to asymmetry, then ($d\theta/ds$ still assumed zero)

$$\frac{d\lambda}{ds} = -\theta - C_2\lambda \tag{8}$$

This equation expresses that, for $\frac{d\theta}{ds} = 0$, the contribution of the asymmetrical strain to $d\lambda/ds$ was $-C_2\lambda$. Also for the symmetrical strain, in which case $\lambda - \frac{d\theta/ds}{K} = 0$ (kinematic shimmy), there was, of course, no contribution due to asymmetry. Assume now a linear interpolation between these two limiting cases. Thus, finally,

$$\frac{d\lambda}{ds} = -\theta - C_2\left(\lambda - \frac{1}{K}\frac{d\theta}{ds}\right) \tag{9}$$

A method of determining C_2 is provided by equation (7) which, when integrated, gives

$$\log_e \frac{\lambda}{\lambda_0} = - C_2(s - s_0)$$

If, with the spindle clamped at $\theta = 0^{\circ}$, the tire is deflected a known amount λ_0 and rolled ahead a known distance and the new λ measured, C_2 may be computed. It was seen earlier that C_2 was of the order of magnitude of $1/r$; that is, it would be of the order of 6 for a 2-inch-radius tire. The constant C_2 was determined for two model tires under different loads and found to be 6.2 and 3.4. Considerable variations of this constant with tire pressure and load have been found.

A method of obtaining the constant known λ necessary for the measurement of C_1 is provided through equation (8). Here it is seen that, if the wheel is pushed along at a constant angle θ, λ will increase (negatively) until $\theta = -C_2\lambda$, in which case $\frac{d\lambda}{ds} = 0$ and equilibrium is reached.

When the wheel is moving ahead at a finite constant velocity, the phenomena represented by equations (6) and (9) occur simultaneously. Therefore, to get the total effect, combine the two equations, thus obtaining

$$\frac{V^2}{C_1}\frac{d^3\theta}{ds^3} + \left(\frac{1}{K} + \frac{C_2}{C_1}V^2\right)\frac{d^2\theta}{ds^2} + \theta = 0 \tag{10}$$

The solution for the natural modes of motion represented by equation (10) is

$$\theta = Ae^{\alpha_1 s} + Be^{\alpha_2 s} + Ce^{\alpha_3 s} \tag{11}$$

where the α's are the three solutions of the so-called auxiliary equation

$$\frac{V^2}{C_1}\alpha^3 + \left(\frac{1}{K} + \frac{C_2}{C_1}V^2\right)\alpha^2 + 1 = 0 \tag{12}$$

One of these α's is real and negative and corresponds to a nonoscillatory convergence. The other roots are conjugate complex numbers and correspond to the shimmy under consideration. The roots will be of the form $a \pm \omega i$. If the divergence a is positive, the oscillation will steadily increase in amplitude (while its amplitude is not large enough for skidding to occur); and, if a is negative, the oscillation will steadily decrease in amplitude and eventually disappear. The meaning of the quantity "divergence" may be illustrated by saying that it is approximately equal to the natural logarithm of the ratio of successive maximum amplitudes to the distance between them. The quantity ω is equal to 2π times the number of oscillations per foot. The frequency therefore is $\omega V/2\pi$. The phase angle is obtained by substituting for θ in equation (6) the value obtained from the foregoing procedure and solving for λ.

The divergence, the frequency, and the phase relations thus derived for typical model tire constants are plotted in figure 29. For small velocities (0 to 6 ft per sec) the frequency corresponds to kinematic shimmy; it is proportional to velocity. The divergence increases rapidly, however, because the spindle angle lags on account of the moment of inertia about the spindle axis, thus allowing more lateral deflection than would occur in a kinematic shimmy. On the next half cycle, a larger spindle angle is reached and the process repeats. As the velocity is further increased, the lag, and hence the asymmetry of the strain, further increase until the strain becomes almost entirely asymmetrical. For this condition, $\frac{d\theta}{ds}\frac{1}{K} \ll \lambda$. Then the restoring torque on the spindle is approximately proportional to λ. (See equation (6).) The tire deflection λ (measured negatively) will, however, still lag somewhat behind θ because, after the wheel is turned through a given angle, a certain forward distance is required for equilibrium tire deflection to be reached. Thus, the restoring force will again lag the displacement. As the velocity increases in the high-velocity range, the frequency stays nearly constant (see fig. 29) and the distance corresponding to a single oscillation increases. Hence this constant lag becomes a smaller part of the cycle and the divergence decreases at high velocities.

It will be appreciated that the foregoing theory considers only the fundamental phenomena taking place in shimmy. Other phenomena occurring simultaneously have been neglected. Some of the more important of the neglected phenomena are:

1. Miscellaneous strains (other than lateral tire deflection) occurring in the tire. A rubber tire being an elastic body will distort in many complicated ways while shimmying. In particular, there will be a twist in the tire due to the transmission of torque from the ground to the wheel.

2. Two effects will cause the stiffness constants of the tire to change with speed. First, centrifugal force on the rubber will make the tire effectively stiffer at high speeds. Second, much of the energy used to deflect the tire will go into compressing the air. The compressibility of the air will change with the speed of compression owing to the different amounts of heat being transferred from it.

3. There will be a gyrostatic torque about the spindle axis caused by the interaction of the rotation of the wheel on the axle and the effective rotation of part of the tire about a longitudinal axis on account of the lateral tire deflection. This torque will later be shown to have a noticeable effect on the results.

The inclusion of items 1 and 2 in the theory would obviously be very difficult. It is therefore necessary to resort to experiment to determine whether the present theory gives an adequate description of the phenomena. If so, the omission of these and any other items will be justified.

An experimental check on the theory was obtained by measuring the divergence and the frequency of the shimmy on the belt machine at two caster angles and at a series of velocities. These measurements were made by placing a lighted flashlight bulb on a 6-inch sting ahead of a model castering wheel with a ball-bearing spindle and then taking high-speed moving pictures of the flashlight bulb with the wheel free. The photographs were made with time recordings on the film, and the belt carried an object that interrupted light from a fixed flashlight bulb and thus recorded the belt speed on the film.

The divergence and the frequency of the shimmy were obtained by measuring the displacements and the times corresponding to successive maximum amplitudes (while the amplitude was still small enough to make all the assumptions valid). The results are plotted in figure 30.

In order to compare these results with the theory, the constants C_1, C_2, and K were determined on the same tire at the two caster angles by the previously described methods. It was found that $C_2 = 6.2$ feet^{-1}; that $K = 62.5$ feet^{-2}; and that, for 5° caster angle, $C_1 = 71{,}100$ feet^{-1}-second^{-2} and, for 20° caster angle, $C_1 = 104{,}000$ feet^{-1}-second^{-2}. The roots of equation (12) were then found and the divergence and the frequency of the shimmy were computed for a series of velocities and at caster angles of 5° and 20°. These results are plotted in figure 31 and, for purposes of comparison, the experimental curves are also reproduced.

The agreement between theory and experiment is considered satisfactory as regards qualitative results. It will be noticed, however, that the theoretical values of the divergence are decidedly too large at high velocities, say 25 feet per second.

Kantrowitz's theory and his experimental results will now be compared with the new theory proposed in the present article:

First of all, the equation of motion (22) of the wheel has for the general solution the function

$$z = C_1e^{s_1t} + C_2e^{s_2t} + C_3e^{s_3t} + C_4e^{s_4t}$$

where C_1, C_2, C_3, C_4 indicate four arbitrary parameters and s_1, s_2, s_3, s_4 the four roots of the characteristic equation

$$\frac{I}{Rav^2} s^4 + \frac{I}{Sav} s^3 + \left(\frac{I\epsilon}{Ta} + \frac{1}{SRav^2} + \frac{a}{TRv^2}\right)s^2 + \frac{a + \epsilon}{TSv} s + \frac{a + \epsilon}{TSa} = 0$$

The coefficients of this equation being always positive, the four roots are real, negative or conjugate negative in groups of two. The negative roots, if existing, correspond to convergent terms and therefore cannot be generators of shimmy; as a result shimmy must arise from the imaginary roots. Assuming now that s_1 and s_2 are two conjugate complex numbers of the form $\lambda \pm \omega i$. The corresponding exponentials are then imaginary and it is advisable to modify the expression. Transformed in trigonometrical terms by means of the Euler formulas, these two exponentials are combined to produce a term of the form $Ae^{\lambda t}\sin(\omega t + \varphi)$, that is an exponential sinusoidal oscillation, that is stable when λ is negative and unstable when λ is positive.

The theoretical study of shimmy can now be successfully completed with the calculation of the variations of λ and ω in terms of the coefficients of equation (22). The same direct method used in the simplified theory for defining the curves of λ and ω in the terms of velocity v could be applied, but the task would be drawn out and difficult. In fact it will be shown that it is not absolutely necessary to interpret the phenomenon of shimmy quantitatively and qualitatively in its minute details. Use will be made of Kantrowitz's theory and experimental data.

We shall first examine what the kinematic shimmy can be. The record of the track of the tire on the ground, the place of the center M of the control surface, of elongation y, was expressed by an equation where y had the same form as equation (22). Replacing z by y in equation (22), and taking the path distance x instead of time t as the independent variable, yields

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = v\frac{dy}{dx} \qquad \frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}$$

$$\frac{d^3y}{dt^3} = v^3\frac{d^3y}{dx^3} \qquad \frac{d^4y}{dt^4} = v^4\frac{d^4y}{dx^4}$$

and equation (22) becomes

$$\frac{Iv^2}{Ra}\frac{d^4y}{dx^4} + \frac{Iv^2}{Sa}\frac{d^3y}{dx^3} + \left(\frac{I\epsilon v^2}{Ta} + \frac{I}{SRa} + \frac{a}{TR}\right)\frac{d^2y}{dx^2} + \frac{a + \epsilon}{TS}\frac{dy}{dx} + \frac{a + \epsilon}{TSa}y = 0 \tag{24}$$

When the velocity v is very small and approaches zero, all the terms containing v disappear and the preceding equation becomes equivalent to

$$(T + Sa^2)\frac{d^2y}{dx^2} + Ra(a + \epsilon)\frac{dy}{dx} + R(a + \epsilon)y = 0 \tag{25}$$

Such an equation represents a damped oscillating motion. The tire track will therefore be a damped sinusoid of equation

$$y = Ae^{-\lambda x}\sin(\omega x + \varphi)$$

with

$$\omega^2 = \frac{4(T + Sa^2)R(a + \epsilon) - R^2a^2(a + \epsilon)^2}{4(T + Sa^2)^2} \tag{26}$$

and

$$\lambda = \frac{Ra(a + \epsilon)}{2(T + Sa^2)} \tag{27}$$

The wave length on the ground, that is the period W, is given by

$$W = \frac{2\pi}{\omega}$$

At zero caster length a

$$\omega^2 = \frac{R\epsilon}{T} \qquad W = 2\pi\sqrt{\frac{T}{R\epsilon}} \qquad \lambda = 0$$

So, if $a = 0$, the damping is zero, that is the tire track on the ground is a sinusoid of wave length $W = 2\pi\sqrt{\frac{T}{R\epsilon}}$. Thus the first point made by Kantrowitz and the phenomenon of kinematic shimmy is explained.

An attempt will now be made to compare the computed wave length with that obtained experimentally by the American writer. To this end the characteristics obtained for the tire studied by Schlippe and Dietrich, that is

$$T = 157 \times 10^{-10} \text{ C.G.S. system}$$

$$S = 3.57 \times 10^{-10} \text{ C.G.S. system}$$

$$R = 1.1 \times 10^{-10} \text{ C.G.S. system}$$

$$\epsilon = 4.35 \text{ cm}$$

are used.

The result is

$$W = 2\pi\sqrt{\frac{157}{1.1 \times 4.35}} = 36 \text{ cm}$$

This wave length is greater than the 19.8, 22.5 and 24.1 cm obtained by Kantrowitz, but that is quite natural; the tire of the German writers was 26 cm in diameter, as against 10.2 cm for the U. S. tire. It will now be attempted to find the effect of the caster length a on this wave length from equation (26). We get

for $a = 0$	$W = 36$ cm
$a = 2$ cm	$W = 34$ cm
$a = 5$ cm	$W = 36$ cm
$a = 6$ cm	$W = 39$ cm
$a = 10$ cm	$W = 74$ cm
$a = 13$ cm	$W = \infty$
$a < 13$ cm	W is imaginary

It is seen that the wave length is practically constant for caster lengths up to about 6 cm, because this wave length passes through a minimum near $a = 2$ cm. This explains Kantrowitz's second remark: "the distance along the belt required for one cycle was also found not to vary much with caster angle or caster length."

Unfortunately the author believed he could conclude immediately that the caster length a could be regarded as being devoid of importance, which was a little hasty and not entirely logical. The essential conclusion of the theory presented here is exactly to the contrary. The effect of the caster length a on the phenomena of shimmy is of primary importance.

Furthermore, the preceding tabulation indicates that wave length W is far from being constant as Kantrowitz's theory assumes; it passes through a minimum only at a small value of caster length a, and as soon as a exceeds 6 cm, W is seen to increase in such a way that $W = 74$ cm for $a = 10$ cm. This increase is speeded up more and more and W approaches infinity when a reaches 13 cm. The wave length for a values above 13 cm is imaginary, that is the track of the tire on the ground is an aperiodic curve. The phenomenon of kinematic shimmy has then completely disappeared.

This fact of the aperiodicity of kinematic shimmy when a exceeds a certain value is absolutely sure. It can be easily checked with a caster fitted with an elastic tire.

Kantrowitz's theory can therefore be only a rather rough theory since its starting point is based on the assumption of constant wave length W, which definitely becomes infinite when a reaches values of the order of these ($a \geqslant 10$ cm). As shown previously, this should eliminate the instability of the oscillations, that is shimmy. In any case, the American author was absolutely unable to reach our significant conclusions regarding the effect of caster length, because his theory starts by refusing a priori to take this parameter into consideration.

To sum up the theory proposed in this report, fits into a theory on kinematic shimmy represented by equation (25) in place of the equation

$$\frac{d^2\theta}{ds^2} + K\theta = 0$$

obtained by making $v = 0$ in Kantrowitz's equation (10), and equation (25) seems to correspond more to reality, because it makes allowance for the variations of wave length W in terms of caster length a.

Lastly, it should be added that equations (25) and (26) indicate that kinematic shimmy is usually a damped oscillation, except when $a = 0$, in which case the damping is zero. The equation

$$\frac{d^2\theta}{ds^2} + K\theta = 0$$

on the other hand, cannot allow for a damping, whose presence is easy to confirm by experiment.

It will next be attempted to compare certain conclusions of Kantrowitz's theory with the corresponding conclusions of the suggested theory. For this we shall try to express the coefficients K, C_1, C_2, beginning with the characteristic coefficients T, D, S, R, ϵ.

<u>Coefficient K</u>

Kantrowitz's differential equation (5) assumes $\theta = A \sin s\sqrt{K}$. The wave length of the kinematic shimmy is then

$$W = \frac{R\epsilon}{T}$$

But the present theory indicates that $W = 2\pi\sqrt{\frac{T}{R\epsilon}}$. Therefore

$$K = \frac{R\epsilon}{T} \tag{28}$$

<u>Coefficient C_1</u>

Kantrowitz's equation (6a) gives $C_1 = \frac{\underline{\underline{M}}}{I\lambda}$, that is with our notations $C_1 = \frac{\underline{\underline{M}}}{I\Delta}$ and since

$$\underline{M} = F(a + \epsilon) \quad \text{and} \quad \Delta = TF$$

we get

$$C_1 = \frac{a + \epsilon}{IT} \tag{29}$$

Coefficient C_2

Kantrowitz's equation (7) expressed in our notation reads

$$\frac{d\Delta}{dx} = -\underline{C}_2\Delta \quad \text{so that} \quad \frac{d\Delta}{dx} = \delta = DF = -D\frac{\Delta}{T}$$

hence

$$\underline{C}_2 = \frac{D}{T} \tag{30}$$

This coefficient C_2 is essentially tied to the drift which the American author implicitly assumes to be instantaneous.

Numerical results

Kantrowitz obtained $W = 24.1$ cm, or $K = \frac{2\pi^2}{W^2}$, hence $K = 0.068$ (C.G.S. system). Formula (29) indicates that C_1 must be dependent in a large measure on caster length a. Kantrowitz in fact noted it and found

for $a = 0.43$ cm $\qquad C_1 = 2.330$ (C.G.S. system)

$a = 1.73$ cm $\qquad C_1 = 3.410$ (C.G.S. system)

Moreover, formula (29) makes it possible to write

$$\frac{0.43 + \epsilon}{1.73 + \epsilon} = \frac{2.330}{3.410} \times \frac{I_{0.43}}{I_{1.73}}$$

This equation permits the computation of the length ϵ by assuming $I_{0.43} = I_{1.73}$, which is approximately true on account of the smallness of a.

Hence

$$\epsilon = 2.4 \text{ cm}$$

For C_2 Kantrowitz obtained $C_2 = 6.2$ feet, or $C_2 = 0.2$ (C.G.S. system).

Calculation of T, D, S, R, and ϵ

Coefficient ϵ has already been defined. The other characteristic coefficients of the tire studied by Kantrowitz can be defined by the formulas (28), (29), and (30), in which the inertia I is involved. Kantrowitz gives $I = 1.06 \times 10^{-4}$ in slugs and feet. The slug is a mass unit of 14.6 kg and 1 foot = 30.48 cm, so that

$$I = 1.440 \quad \text{C.G.S. units}$$

Formula (29) gives then

$$T = \frac{a + \epsilon}{I\underline{C}_1} = \frac{0.43 + 2.4}{1.440 \times 2330}$$

or

$$T = 84 \times 10^{-8} \text{ in C.G.S. units}$$

Formula (28) gives

$$R = \frac{KT}{\epsilon} = \frac{0.068 \times 84 \times 10^{-8}}{2.4}$$

or

$$R = 2.38 \times 10^{-8} \text{ in C.G.S. units}$$

likewise formula (30) gives

$$D = \underline{C}_2 T = 0.2 \times 84 \times 10^{-8}$$

or

$$D = 16.8 \times 10^{-8} \text{ in C.G.S. units}$$

Lastly,

$$S = 7 \times 10^{-8} \text{ in C.G.S. units}$$

by reason of the formula $S\epsilon = D$.

The kinematic shimmy of Kantrowitz

By the use of the derived characteristic coefficients together with formula (26) the determination of the wave lengths W of kinematic shimmy made on the Schlippe-Dietrich tire can be applied to Kantrowitz's tire.

For	$a = 0$	$W = 24$ cm
	$a = 2$ cm	$W = 21$ cm
	$a = 4$ cm	$W = 27$ cm
	$a = 6$ cm	$W = 38$ cm
	$a = 11$ cm	$W = \infty$
	$a > 11$ cm	W is imaginary

This tabulation proves that the conclusions valid for the German tire are even more valid for the American tire.

Inversion velocity u

Having defined the characteristic coefficients the inversion velocity u defined previously by the equation

$$u^2 = \frac{a}{IR}$$

can now be computed.

Since a high caster angle is likely to modify this u, we shall not make the calculation for the test series for which this high caster angle was 20° but only for the series where it was 5°, which is small enough to be regarded as zero. Then

$$u^2 = \frac{0.43}{1.440 \times 2.38 \times 10^{-8}}$$

hence

$$u = 112 \text{ cm/sec}$$

that is

$$u = 3.70 \text{ ft/sec}$$

But on considering figure 30 and the curve A of the convergence (for a caster angle of 5° and $a = 0.43$ cm) it is plain that the extension of the experimental curve does not pass through the origin. This circumstance is also relevant for the curves of figures 20 and 21 established by Wylie. The extension of the experimental curve A plotted by Kantrowitz exactly intersects the axis of the velocities in the axis of the abscissa $v = 3.7$ ft/sec; as stipulated by the theory.

In contrast to Kantrowitz's theory, this is an additional accomplishment of the present theory. In fact, it will be shown that by Kantrowitz's equation (10) the curve $\lambda(v)$ passes through the origin which does not correspond at all to the experimental results.

In order to make the comparison between the two theories more accurate, consider Kantrowitz's equation (10) and compare it with equations (6) and (22) into which the suggested theory fits. First of all, equation (10) is transformed by taking the time t as independent variable in place of the distance s, and then the amplitude z computed in place of the angular elongation θ. On the one hand

$$\theta = \frac{z}{a} \qquad \frac{d\theta}{ds} = \frac{1}{a}\frac{dz}{ds} \qquad \frac{d^2\theta}{ds^2} = \frac{1}{a}\frac{d^2z}{ds^2} \qquad \frac{d^3\theta}{ds^3} = \frac{1}{a}\frac{d^3z}{ds^3}$$

on the other

$$s = vt \qquad \frac{dz}{ds} = \frac{1}{v}\frac{dz}{dt} \qquad \frac{d^2z}{ds^2} = \frac{1}{v^2}\frac{d^2z}{dt^2} \qquad \frac{d^3z}{ds^3} = \frac{1}{v^3}\frac{d^3z}{dt^3}$$

so that equation (10) becomes

$$\frac{1}{\underline{C}_1 v}\frac{d^3z}{dt^3} + \left(\frac{1}{Kv^2} + \frac{\underline{C}_2}{\underline{C}_1}\right)\frac{d^2z}{dt^2} + z = 0$$

that is

$$\frac{IT}{(a + \epsilon)v}\frac{d^3z}{dt^3} + \left(\frac{ID}{a + \epsilon} + \frac{T}{R\epsilon v^2}\right)\frac{d^2z}{dt^2} + z = 0 \tag{31}$$

by expressing the constants K, C_1, C_2 by equations (28), (29), and (30). We repeat equation (6) of the simplified theory

$$\frac{IT}{(a + \epsilon)v} \frac{d^3z}{dt^3} + \frac{ID}{a + \epsilon} \frac{d^2z}{dt^2} + \frac{a}{v} \frac{dz}{dt} + z = 0 \tag{6}$$

and equation (22) of the complete theory after multiplying by $\frac{TSa}{a + \epsilon}$, taking into account the relationship $S\epsilon = D$

$$\frac{ITS}{R(a + \epsilon)v^2} \frac{d^4z}{dt^4} + \frac{IT}{(a + \epsilon)v} \frac{d^3z}{dt^3} + \left[\frac{ID}{a + \epsilon} + \frac{T + Sa^2}{R(a + \epsilon)v^2}\right] \frac{d^2z}{dt^2} + \frac{a}{v} \frac{dz}{dt} + z = 0 \tag{22}$$

Now equation (31) can be compared with equations (6) and (22). Incidentally, the turn coefficient R (consequently the phenomenon of torsion) was implicitly introduced in Kantrowitz's theory, although its author claims to have ignored the torsion of the tire. Kinematic shimmy can only be explained by the effect of this torsion and it is, in fact, easy to prove that equation (6), which would be a complete theory if this phenomenon of torsion could be disregarded, cannot give an account of kinematic shimmy. A second preliminary remark is needed. When the linear differential equation of the third order with positive coefficients

$$Az''' + Bz'' + Cz' + D = 0$$

is considered, Routh's rule is summed up as follows: for the oscillation to be convergent it is sufficient and necessary that $BC > AD$. In equation (31) $C = 0$; therefore the oscillation can never be convergent. As a result Kantrowitz could not be led to suspect the possibility of combating shimmy by a simple judicious combination of parameters. This was unfortunate since it led to the design of landing gears having as short a caster length as possible. Lastly, in comparing equation (6) of the simplified theory with equation (22) of the complete theory it is seen that the only difference is the presence in equation (22) of two supplementary terms:

$$\frac{ITS}{R(a + \epsilon)v^2} \frac{d^4z}{dt^4} \quad \text{and} \quad \frac{T + Sa^2}{R(a + \epsilon)v^2} \frac{d^2z}{dt^2}$$

Each one of these terms carries the factor v^2 in the denominator. So, when the velocity increases infinitely, the motion represented by equation (22) tends asymptotically towards the motion represented by equation (6). In other words, when the velocity is large enough the two equations (6) and (22) can be regarded as practically equivalent. So, for a comparison of equations (22) and (31) at high velocities, it is sufficient to compare equations (6) and (31).

We shall now define the relationship existing between velocity v and divergence λ in the case of equation (31) (that is, in the Kantrowitz theory), as we did before for equation (6).

Employing the same method as for equation (11), the system

$$\frac{IT}{(a + \epsilon)v} \lambda(\lambda^2 - 3\omega^2) + \left(\frac{ID}{a + \epsilon} + \frac{T}{R\epsilon v^2}\right)(\lambda^2 - \omega^2) + 1 = 0$$

$$\frac{IT}{(a + \epsilon)v}(3\lambda^2 - \omega^2) + \left(\frac{ID}{a + \epsilon} + \frac{T}{R\epsilon v^2}\right)2\lambda = 0$$

is obtained. To obtain the relationship looked for between v and λ, simply eliminate ω^2 from the two preceding equations. Thus

$$\frac{8IT}{(a + \epsilon)v} \lambda^3 + 8\left(\frac{ID}{a + \epsilon} + \frac{T}{R\epsilon v^2}\right)\lambda^2 + \frac{2(a + \epsilon)v}{IT}\left(\frac{ID}{a + \epsilon} + \frac{T}{R\epsilon v^2}\right)^2 \lambda = 1 \quad (32)$$

This equation assumes that time t is the independent variable. Now, in his calculations as well as in the curves of figures 30 and 31, Kantrowitz has used the distance $x = vt$ as the independent variable. The fundamental oscillation of shimmy

$$z = e^{\lambda t} \sin \omega t$$

is then

$$z = e^{\mu x} \sin \frac{\omega}{v} x$$

by putting

$$\mu = \frac{\lambda}{v}$$

To obtain the equation giving the new expression of the divergence, λ is simply replaced by $v\mu$ in equation (32). We then get, after arrangement of the terms of this equation

$$\frac{2I\mu}{a + \epsilon}\left(2\mu + \frac{D}{T}\right)^2 v^4 + \left(\frac{8}{R\epsilon}\mu^2 + \frac{4D}{TR\epsilon}\mu - \frac{1}{T}\right)v^2 + \frac{2(a + \epsilon)}{IR^2\epsilon^2}\mu = 0$$

With respect to velocity v this is an equation of the fourth power, whose two primary roots, if they are real, have the same sign. In order that these roots are real, the discriminant must be positive. Therefore after some simplifications we obtain

$$16T\mu^2 + 8D\mu - R\epsilon \leqslant 0$$

which is true when μ has a value between μ_1 and μ_2 of this trinomial. These roots being of opposite sign, it is sufficient that μ is less than the positive root, hence

$$\mu < \frac{\sqrt{D^2 + TR\epsilon} - D}{4T}$$

It is readily apparent that, when this condition exists, the roots are, of necessity, positive, because the coefficient of v^2 is then negative. In fact, this coefficient is a trinomial of the second degree in μ allowing two roots of opposite sign, hence the trinomial is negative when μ is less than the positive root. But this root is

$$\mu_1 = \frac{\sqrt{D^2 + 2TR\epsilon} - D}{4T}$$

and it is obviously greater than the positive root of the discriminant. Thus the first condition is sufficient to assure the second, $\mu < \mu_1$. The result is that the divergence μ passes through a maximum

$$\mu_m = \frac{\sqrt{D^2 + TR\epsilon}}{4T}$$

for a given velocity v_m given by the equation

$$v_m^2 = \frac{a + \epsilon}{IR\epsilon\left(2\mu + \frac{D}{T}\right)}$$

Thus, for the tire studied by Kantrowitz (curve A in fig. 36)

$$\mu_m = 0.0322 \quad \text{C.G.S. units}$$

$$v = 364 \text{ cm/sec}$$

or in English units of feet (30.5 cm)

$$\mu_m = 0.98 \quad \text{English units}$$

$$v_m = 11.9 \text{ ft/sec}$$

<u>Shape of curve $\lambda(v)$ at high velocities</u>

The variation of $\lambda(v)$ when the velocity increases infinitely, will be examined by putting $\lambda = \frac{K}{v}$ and trying to define K in terms of the characteristic coefficients. Entering this value in the preceding equation while ignoring the infinitely small terms gives the relationship

$$\frac{2(a + \epsilon)v}{IT}\left(\frac{ID}{a + \epsilon}\right)^2 \frac{K}{v} = 1$$

so that

$$K = \frac{T(a + \epsilon)}{2D^2I}$$

The same calculation with equation (11) gives

$$K' = \frac{(T - aD)(a + \epsilon)}{2D^2I}$$

Thus it is seen that the present theory produces lower values for λ at high velocities than Kantrowitz's theory and seems to explain to some extent the remark made by him that the values deduced by his formula are clearly too high for high velocities.

To complete the comparison of Kantrowitz's theory, of the preceding theory and of the experimental results, the calculations of λ and μ at various velocities were made

(1) By Kantrowitz's theory (equations (32) and (32a))

(2) By the present theory in its approximate form (equation (12))

(3) By the present theory in its complete form (equation (22))

	Kantrowitz's theory			Present theory			
				Approximate form	Complete form		
v cm/sec	λ C.G.S.	μ C.G.S.	μ English unit	λ C.G.S.	λ C.G.S.	μ C.G.S.	μ English unit
0	0			0			
100	.87	0.0087	0.265				
112					0	0	0
200	4.77	.0238	.730		10.5	.0525	1.6
294				λ_m = 24.7			
300	9.36	.0312	.95		19.5	.0650	1.98
364	11.7	.0322	.98				
400	12.7	μ_m.0318	μ_m.97		20.6	.0515	1.57
500	14.8	.0296	.90		20.8	.0415	1.27
600	16.0	.0267	.815		20.8	.0346	1.05
700	16.6	.0238	.725				
800	16.71	.0209	.635		19.6	.024	.73
900	16.65	.0185	.555				
1000	16.4	.0164	.50		17.6	.0176	.54
1500	13.62	.0091	.28		15.3	.0102	.31
2000	11.48	.00575	.175		11.3	.00565	.17
2500	10	.004	.12		10	.004	.12

The table is graphically represented in figure 32 with μ plotted against the velocity v for

(1) Experimental data (curve E)

(2) Data from Kantrowitz theory (curve K)

(3) Data from present theory in its complete form (curve C)

The velocity is expressed in cm/sec and μ in English units used by Kantrowitz. To obtain curve E, plot the 15 experimental points given by Kantrowitz, then draw the most likely mean curve defined by those 15 points.

The plotting of curve K involves the resolution of a linear differential equation of the third order which presents no special difficulties, as the necessary calculations can be made by any of several classical methods; the plotting of curve C reduces to resolving equation (22) which is a linear differential equation of the fourth order. It is a more difficult problem and involves considerable paper work, so a graphical method of producing quick and excellent results is desired.

Graphical method for solving linear and homogeneous linear equations

The equation (22) that is to be solved, is of the form

$$Az^{IV} + Bz^{III} + Cz^{II} + Dz^{I} + Ez = 0$$

which, putting $z = Ke^{ext}$, gives the characteristic equation

$$f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

Sought are the real numbers or, complex x, solutions of this equation. Consider, therefore, the complex plane O, X, Y, (fig. 33). To every point m of the prefix x the preceding equation makes correspond a point d' with prefix f(x). This point M is easily obtained from point m by simple graphical operations with rule and dividers. Point m corresponds to a root x when M reaches O. Then consider a point m' near m to which corresponds a point M' near M, and which gives by definition

$$\frac{\overrightarrow{MM'}}{\overrightarrow{mm'}} = f'(x)$$

However, the value of $f'(x)$ is not dependent on the position of m'; but solely on that of m. Therefore, if two vectors $\overrightarrow{Pm''}$ and PM'' equipollent to $\overrightarrow{mm'}$ and MM' are involved, the triangle Pm''M'' remains similar regardless of the position of point m' near m. Consequently, in the only condition where the prefix x of m is not one of the roots of the equation $f'(x) = 0$ (which would be an exceptional case since it is impossible to have more than three points in the entire plane satisfying $f'(x)$) a point m' near m is, as a rule, easily found, so that its transformed M' is closer to O than point M. After a sufficient number of operations, the point M can be gradually brought as near to point O as desired. This is the principle of a graphical solution of algebraic equations by successive approximations.

It should be noted in passing that the remark contains implicitly the rigorous proof of d'Alembert's theorem which states that any algebraic equation with real or imaginary coefficients has always at least one real or imaginary root. In many cases the foregoing operations can be made to converge very quickly. The method consists in taking as point m the second point m_1 so that

$$\frac{\overrightarrow{MO}}{\overrightarrow{mm_1}} = f'(x) = \frac{\overrightarrow{MM'}}{\overrightarrow{mm'}}$$

M and M' being constructed on the line of m and m', only m' being chosen near m on the radius OM. The graphical method of defining the point m_1 by utilizing the ratio $\frac{\overrightarrow{MM'}}{\overrightarrow{mm'}}$ is much more simple than trying to determine m_1 directly from the derivative $f'(x)$. After obtaining m_1 deduce M_1, then begin again with torque m_1M_1, the operations to be made on torque mM. The result is a torque m_2M_2. As a rule, the successive points $M, M_1, M_2, \ldots$ come closer and closer to O and the points $m, m_1, m_2, \ldots$ tend to a limiting point whose prefix $\lambda + i\omega$ is the root of the equation $f(x) = 0$. The function

$$z = e^{\lambda}\sin \omega t$$

is then the corresponding solution of the differential equation that was to be solved.

The unusual feature of this method, whose principle is apparent from Newton's method for finding real roots of algebraic equations is the small number of operations necessary to obtain convergence as soon as a root is approached. This graphical method is absolutely general and can be applied to algebraic equations of any degree.

After this digression the study of the three curves in figure 32 is resumed.

The comparison of the experimental curve E with Kantrowitz's theoretical curve K has shown that the maximum of the latter curve is not clearly enough indicated. Curve K is much too flat and drops too slowly when the velocity increases. Specifically curve K gives slightly too small divergences near the maximum and by way of contrast, definitely too great divergences at high velocities, as the author himself admits.

The comparison of the experimental curve E with curve C given by the present theory indicates that the latter gives value sensibly too great, but that the general shape of the curve C (existence of the same inversion velocity u, acuteness of the maximum, rapidity of decrease at increasing velocity) approaches curve E' more closely than the flat shape of curve K. The correspondence between curve C and curve E is almost a simple expansion of the ordinates in a ratio approaching 1 as the velocity increases. So in spite of the similarity of forms between E and C there still is a certain discrepancy between the experimental and theoretical results of the present theory. It is the first discrepancy of this kind that has been found. A careful examination to see whether this can be explained within the framework of the present theory, was therefore indicated.

It seems preferable to have the theoretical discrepancies on the high side rather than the low (as in the case of curve K in the neighborhood of the maximum). The divergence of the oscillations corresponds, in fact, to a storage of mechanical energy in the castering wheel and the aim of the theory of shimmy is precisely to discover the source and the laws of this energy. Shimmy is accompanied by various energy dissipating phenomena for which the theory makes no allowance, such as friction in the castering axis, the slight slippage of the tire close to the periphery of the contact surface and particularly of the tail point, the reactive effects at the inside of the tire, etc. The last cause is far from being negligible: according to the general scheme of oscillatory motions shimmy is accompanied by a periodic and continuous transformation of kinetic energy of the wheel into potential stress energy of the tire and vice versa. It is known that a stressed tire recovers only part of the stress energy. Drop tests on tire indicate, for example, the height of rebound is always from 15 to 30 percent less than the height of drop, the loss being due largely to the reaction effects inside the tire. It is therefore to be expected that in a correct theory of shimmy the values of divergence are always greater than the experimentally observed values. Still, this explanation hardly justifies a difference exceeding 10 to 20 percent. The difference between the maximum point of the theoretical curve C and the experimental point amounts to $\frac{8}{20}$ = 40 percent. There must be some other cause for explaining such a marked discrepancy. There is only one possibility, namely, the numerical values used for T, D, S, R, and ϵ are probably not rigorously exact. Unfortunately these factors had been defined indirectly from Kantrowitz's factors K, C_1, and C_2, hence there is a possibility that some of the defined factors, T, D, S, R, and ϵ were only approximate.

This matter will be discussed in a little greater detail: Formulas (28), (29), and (30) were taken as a starting point. Formula (29) resulted in

$$\frac{0.43 + \epsilon}{0.73 + \epsilon} = \frac{2330}{3410} \frac{I_{0.43}}{I_{1.73}}$$

which made it possible to compute $\epsilon = 2.4$.

Obviously, this value for ϵ is only approximate, and for two reasons: first, it was assumed that

$$I_{0.43} = I_{1.73}$$

although the second of these values is certainly greater than the first. On the other hand, formula (29) makes no allowance for inclination of caster. Rigorously written it should have been

$$\underline{C}_1 = \frac{M}{I\Delta} = \frac{F(a + \epsilon) + \rho\theta}{I\Delta}$$

with $\Delta = TF$ and $\theta = DF$, that is

$$\underline{C}_1 = \frac{a + \epsilon}{IT} + \frac{\rho D}{IT}$$

the parameter ρ being defined later. However, T was computed by formula (29) from the derived value for ϵ, which is the reason that the value obtained for this parameter cannot be entirely rigorous. Then R was computed by formula (28). This formula, utilizing the length of kinematic shimmy, seems to be rigorous. Unfortunately, it contains ϵ and T which limits the accuracy to be expected from R. Lastly, D was computed by formula (30) and S by the relation $S\epsilon = D$. Our formula (30) is simply Kantrowitz's formula (7), which is equivalent to assuming that drift is instantaneous. And this assumption is, as already shown in section III-B, only a simplifying assumption frequently permitted, but which, in this instance, is likely to taint the calculation of $\underline{C}_2$ with an error.

A rigorous determination of the equation of the curve studied by Kantrowitz in figure 28 will now be attempted. This curve was used to determine the coefficient $\underline{C}_2$.

Let y represent the distance of the center M of the contact tread with respect to the undeflected plane of the wheel. With our usual notations (fig. 34)

$$\frac{d^2y}{dx^2} = -R\underline{M}$$

$$\underline{M} = \frac{\alpha}{S} - F\epsilon$$

$$\alpha = \frac{dy}{dx}$$

$$F = \frac{1}{T}(z - y) = -\frac{y}{T}$$

So the final equation after eliminating $\underline{M}$, α, and F from the preceding equations reads

$$\frac{1}{R}\frac{d^2y}{dx^2} + \frac{1}{S}\frac{dy}{dx} + \frac{\epsilon}{T} y = 0$$

The most general solution of this differential equation is of the following form

$$y = Ae^{s_1 x} + Be^{s_2 x}$$

A and B are two coefficients depending on the initial conditions and s_1 and s_2 the two roots of the equation of the second degree

$$\frac{s^2}{R} + \frac{s}{S} + \frac{\epsilon}{T} = 0$$

Therefore, the equation of curve L, which represents the path of the center M of the contact tread (fig. 34) is

$$y = Ae^{s_1 x} + Be^{s_2 x} \quad \text{and not} \quad y = Ce^{-c_2 x} = Ce^{-\frac{D}{T}x}$$

as assumed by the American author without adequate reason. As a consequence the parameter D computed by wrongfully assuming that it is the equation of curve L, may very well be tainted with an appreciable error, which by itself might be sufficient to explain the discrepancy between curve E and C in figure 32.

V. EFFECT OF RESTORING TORQUE (Inclination of Castering Axis)

EFFECT OF FLUID AND SOLID FRICTION (Damping of Shimmy)

A. Effect of Restoring Torque

In the foregoing it has been assumed that the castering axis was vertical. The next logical question is: what is the effect of an inclination of the castering axis on the stability of the vibrations? It is readily seen that, provided it is not too great, such an inclination should be equivalent to the existence of a restoring torque proportional to the angle θ, this torque being moreover, negative in the sense of figure 25, that is tending to move the wheel from its mean position $\theta = 0$.

Now the general problem of the effect of a restoring torque $\underline{C}_r = \rho\theta$ will be analyzed. To express the problem in equation form, the same line of reasoning used for the system of four equations (16), (17), (18), and (19) is followed. It is apparent that, in these conditions, equations (16), (17), and (19) are not changed by the existence of a restoring torque. Equation (18) can be replaced by

$$I \frac{d^2\theta}{dt^2} = -F(a + \epsilon) - \underline{M} - \rho\theta$$

or

$$\frac{I}{a} \frac{d^2z}{dt^2} = -F(a + \epsilon) - \underline{M} - \frac{\rho}{a} z$$

since $\theta = \frac{z}{a}$. To the moments $-F(a + \epsilon)$ and $-\underline{M}$ produced by the reactions of the ground on the contact tread, must be added the moment $-\rho\theta$ of the restoring torque, ρ being positive when a true restoring torque is involved and negative when a torque of opposite effect is involved (as for example in the case of a disposition of the caster as illustrated in fig. 25).

The result is the equation system

$$\frac{1}{v^2}\frac{d^2y}{dt^2} = -R\underline{M} \tag{16}$$

$$z - y = TF \tag{17}$$

$$\frac{I}{a}\frac{d^2z}{dt^2} = -F(a + \epsilon) - \underline{M} - \frac{\rho}{a} z \tag{18a}$$

$$\underline{M} = \frac{1}{Sv}\frac{dy}{dt} + \frac{z}{Sa} - F\epsilon \tag{19}$$

The values of F and $\underline{M}$ of equations (17) and (19) entered in equations (16) and (18a) give the two equations

$$\frac{1}{Rv^2}\frac{d^2y}{dt^2} + \frac{1}{Sv}\frac{dy}{dt} + \frac{\epsilon}{T} y + \left(\frac{1}{Sa} - \frac{\epsilon}{T}\right)z = 0 \tag{20}$$

$$\frac{1}{Sv}\frac{dy}{dt} - \frac{a}{T} y + \frac{I}{a}\frac{d^2z}{dt^2} + \left(\frac{\rho}{a} + \frac{1}{Sa} + \frac{a}{T}\right)z = 0 \tag{21a}$$

The values of z from equation (20) posted in equation (21a) give the differential equation

$$\frac{I}{Rav^2}\frac{d^4y}{dt^4} + \frac{I}{Sav}\frac{d^3y}{dt^3} + \left(\frac{I\epsilon}{Ta} + \frac{1}{SRav^2} + \frac{a}{TRv^2} + \frac{\rho}{Rav^2}\right)\frac{d^2y}{dt^2} +$$

$$\left(\frac{a + \epsilon}{TSv} + \frac{\rho}{Sav}\right)\frac{dy}{dt} + \left(\frac{a + \epsilon}{TSa} + \frac{\epsilon\rho}{aT}\right)y = 0 \tag{33}$$

It is easily seen that the second remark made below equation (22) is still applicable here. Therefore, to obtain the differential equation of motion of the wheel y is simply replaced by z in the preceding equation. The result is, then, a linear differential equation of the fourth order of the form

$$a_0 z^{IV} + a_1 z^{III} + a_2 z^{II} + a_3 z^{I} + a_4 z = 0$$

In connection with equation (22) it was shown that Routh's conditions for stable oscillation are as follows

$$a_1 a_2 > a_0 a_3$$

$$a_1 a_2 a_3 > a_0 a_3^2 + a_1^2 a_4$$

Applied to equation (33) the first of these conditions gives, after reductions

$$\frac{I\epsilon}{a} + \frac{T}{SRav^2} > \frac{\epsilon}{Rv^2}$$

or, if the fundamental relation $S\epsilon = D$ is taken into account

$$v^2 > \frac{aD - T}{IRD} \tag{34}$$

The second of the preceding equations gives, after reductions

$$\frac{I\epsilon}{Ta} + \frac{1}{SRav^2} + \frac{\rho T}{SRa^2v^2(a + \epsilon)} > \frac{\epsilon}{TRv^2} + \frac{I}{Sa^2} + \frac{\epsilon\rho}{Rav^2(a + \epsilon)}$$

or

$$\left(IRv^2 - a - \frac{\rho T}{a + \epsilon}\right)(S\epsilon a - T) > 0$$

that is if the fundamental relationship is taken into consideration

$$\left(IRv^2 - a - \frac{\rho T}{a + \epsilon}\right)(aD - T) > 0 \tag{34a}$$

To complete the discussion of these stability conditions, two cases are distinguished according to the sign of the factor ρ.

(a) When a true restoring torque is involved, that is when $\rho > 0$, the discussion presented following equations (23) and (23a) applies here also. Consider the velocity u defined by the equality

$$IRu^2 = a + \frac{\rho T}{a + \epsilon}$$

First case: $aD > T$. The condition (34a) is proved when $v > u$.

Second case: $aD < T$. The condition (34a) is proved when $v < u$.

It is plain that in either case, when equation (34a) is proved equation (34) is also confirmed. Therefore, the conditions of stability of oscillation are

First case: $aD > T$ with $v > u$

Second case: $aD < T$ with $v < u$

The problem therefore is like that discussed following equation (22) and it is clear that the effect of a true restoring torque on the stability conditions is simply an increase in the inversion velocity u.

(b) When a negative torque is involved, that is when ρ is negative, which is reached at an inclination of the caster as shown in figure 25, it can no longer be asserted that the confirmation of equation (34a) automatically entails the confirmation of equation (34).

It is seen that, if $aD < T$ this condition (34) is always proved, but if $aD > T$, the condition (34) is proved only when $v > w$, w denoting the velocity defined by the equality

$$IRw^2 = a - \frac{T}{D}$$

The stability conditions of the oscillation are therefore

First case: $aD > T$ with $v > u$ and simultaneously $v > w$

Second case: $aD < T$ with $v < u$

On trying to compare u and w it is readily apparent that in order that $u > w$ it is necessary and sufficient that $-\rho > \frac{a + \epsilon}{D}$. From the previously defined values for ϵ and D and the value of ρ that will be computed subsequently (formula 35) it follows that, except for large inclination of the axis and substantial wheel loads, $u > w$, regardless of the caster length a.

The discussion is again similar to that relevant to equation (22) to the effect that a negative restoring force on the stability conditions is simply a decrease of the inversion velocity u.

Let us now refer to figure 30 and compare the experimental curves A and B given by Kantrowitz. By extending these curves up to their intersection point with the velocity axis the corresponding inversion velocities are obtained. Obviously the inversion velocity relative to curve B is substantially lower than that with respect to curve A. But curve B is exactly that which has been obtained for a 20 degree inclination of the caster axis, corresponding to a negative restoring torque. This experimental check although purely qualitative (because the author failed to mention the load applied to the wheel, and which is needed for defining the factor ρ) is an added proof of the present theory.

The decrease in u following the inclination of the caster axis is much more evident in this example since the experiments relative to the curve B of figure 30 had been made with a caster length ($a = 1.73$ m) greater than that relative to curve A ($a = 0.43$ m). So without inclination the inversion velocity would have been greater, and it actually would have resulted in

$$u = \sqrt{\frac{a}{IR}} = \sqrt{\frac{1.73 \times 10^8}{1.440 \times 2.38}} = 225 \text{ cm/sec}$$

for curve B, instead of $u = 112$ cm/sec for curve A.

Calculation of coefficient ρ for an inclination φ of the spindle axis.

For the case of an inclination φ of the spindle axis, the coefficient ρ of the negative restoring torque can be defined in terms of the load P of the wheel on the ground, the caster length a, and the angle φ. Consider figure 35 which represents a tire touching the ground at M and assume that AP is the spindle axis. When the wheel turns about AP it can be assumed for first approximation that M describes a circle centered on AP, that is a circle of center H. Axis AP is assumed to be stationary while the wheel turns through a small angle θ around this axis. Now it is readily seen that the point M is raised up to a height

$$h = \overline{MH} \frac{\theta^2}{2} \sin \varphi = a \frac{\theta^2}{2} \sin \varphi \cos \varphi$$

The work performed by the force P is therefore

$$Ph = Pa \frac{\theta^2}{2} \sin \varphi \cos \varphi$$

with $\underline{C}_r = \rho\theta$ representing the negative restoring torque it is evident that the foregoing work is equal to that performed by the torque $\underline{C}_r$, that is

$$\int_0^\theta \underline{C}_r \, d\theta = \int_0^\theta \rho\theta \, d\theta = \rho \frac{\theta^2}{2}$$

Comparison of this expression with the preceding one, gives the sought for expression of the coefficient ρ

$$\rho = -Pa \sin \varphi \cos \varphi \tag{35}$$

So, if the American author had not forgotten to give us the pressure P of the wheel on the ground, we would have been able to use his experimental results for a numerical check of the present theory. In the absence of these data the foregoing considerations can, however, still be utilized for a numerical calculation by adopting an inverse method of looking for the evaluation of P from an examination of the curve B.

From an examination of the curve B in figure 30 it is found that the inversion velocity u is reduced to a value approaching zero, by the inclination of the spindle axis. Hence we assume $u = 0$ It was seen that

$$u^2 = \frac{a}{IR} + \frac{\rho T}{(a + \epsilon)IR}$$

hence

$$\frac{a}{IR} - \frac{Pa \sin \varphi \cos \varphi T}{(a + \epsilon)IR} = 0$$

consequently

$$P = \frac{a + \epsilon}{T \sin \varphi \cos \varphi} = \frac{1.73 + 2.4}{84 \times 10^{-8} \times 0.342 \times 0.94} = 153 \times 10^5 \text{ dynes}$$

that is

$$P \# 15.3 \text{ kg weight}$$

So, for the first series of experiments (fig. 30, curves A and A') we get

$$\rho = -10^7 \times 0.43 \times 0.087 \times 0.996 \# - 3.5 \times 10^5 \text{ C.G.S.}$$

and for the second (curves B and B')

$$\rho = -10^7 \times 1.73 \times 0.342 \times 0.94 \# - 55 \times 10^5 \text{ C.G.S.}$$

Comparative study of the curves of figure 30.

Curve B presents a slight deviation with respect to curve A. It is logical to hunt for an explanation for this different shape with the aid of the present theory. To start from the complete theory, that is from equation (33) would involve considerable paper work. Recourse is therefore had to the simplified theory which obviously produces the equation

$$\frac{I}{Sav}\frac{d^3y}{dt^3} + \frac{I\epsilon}{Ta}\frac{d^2y}{dt^2} + \frac{a + \epsilon}{TSv} + \frac{\rho}{Sav}\frac{dy}{dt} + \frac{a + \epsilon}{TSa} + \frac{\epsilon\rho}{aT}y = 0 \qquad (35)$$

obtained from equation (33) by disregarding all the terms having R in the denominator. The same method that produced equation (11) gives, therefore, when starting from equation (35), the system

$$2T\lambda(\lambda^2 + \omega^2) + vD(\lambda^2 + \omega^2) - \frac{v(a + \epsilon + D\rho)}{I} = 0$$

$$T(3\lambda^2 - \omega^2) + 2vD\lambda + \frac{a(a + \epsilon) + T\rho}{I} = 0$$

The second of these equations gives

$$\omega^2 = 3\lambda^2 + 2\,\frac{vD\lambda}{T} + \frac{a(a + \epsilon)}{IT} + \frac{\rho}{I} \tag{36}$$

which, substituted in the first equation, gives

$$(2T\lambda + vD)\left[4\lambda^2 + 2\,\frac{vD\lambda}{T} + \frac{a(a + \epsilon)}{IT} + \frac{\rho}{I}\right] - \frac{v(a + \epsilon + D\rho)}{I} = 0$$

that is after transformations

$$2D^2I\lambda v^2 + \left[8DTI + (aD - T)(a + \epsilon)\right]v + 2T\lambda\left[4IT\lambda^2 + a(a + \epsilon) + T\rho\right] = 0 \tag{37}$$

Compared to equation (12) this equation differs only by the presence of coefficient ρ in the constant term. It is seen that for a real value of λ there always are real values of v, provided only that the determinant is positive, that is that

$$16DT^2I(\lambda^2 + D\rho) < (a + \epsilon)(aD - T)^2$$

The maximum value λ_m is then

$$\lambda_m{}^2 = \frac{(a + \epsilon)(aD - T)^2}{16DT^2I} - D\rho \tag{38}$$

So, for the curve A ($a = 0.43$ cm, $\rho = -3.5 \times 10^5$ C.G.S. system)

$$\lambda_m = 21.47$$

and for the curve B ($a = 1.73$, $\rho = -55 \times 10^5$ C.G.S. system)

$$\lambda_m = 21.38$$

Thus we obtain two practically equal values, which is in good agreement with the shape of curves A and B.

We shall now turn to the calculation of the corresponding velocities v_m. Since the discriminant of equation (37) is zero in this case,

$$v_m^2 = \frac{T}{D^2 I}\left[4IT\lambda^2 + a(a + \epsilon) + T\rho\right]$$

which gives

$$v_m = 255 \text{ cm/sec} = 8.4 \text{ ft/sec} \quad \text{for curve A}$$

$$v_m = 314 \text{ cm/sec} = 10.3 \text{ ft/sec} \quad \text{for curve B}$$

These values are almost identical with the experimental values in figure 30. But above all this calculation has proved that the present theory allows for the deviation of curve B relative to curve A, which consists essentially of a displacement of the maximum and the adjacent parts toward the higher values of the velocity.

Equation (36) enables the values of the angular frequency ω corresponding to the maximum λ_m to be computed and consequently also the values of the frequency $n = \frac{\omega}{2\pi}$. Thus

$$n = 10.5 \text{ cycles/sec} \quad \text{for curve A'}$$

$$n = 12.4 \text{ cycles/sec} \quad \text{for curve B'}$$

These values very closely approximate those of the experimental curves A' and B' of figure 30. It should be remembered above everything else that the present theory justifies a higher frequency at corresponding velocities for curve B' which is plainly indicated by curves A' and B'.

The considerations are completed by the calculation of the limit of the frequency N when the velocity increases infinitely. Concerning equation (33), it is readily apparent that, when v increases infinitely, the majority of its terms approach zero, so that at the limit this equation is reduced to

$$\frac{I\epsilon}{Ta}\frac{d^2y}{dt^2} + \left(\frac{a + \epsilon}{TSa} + \frac{\epsilon\rho}{aT}\right)y = 0$$

This equation represents a sinusoidal function of the frequency ω hence

$$\omega^2 = \frac{1}{I}\left(\frac{1}{S} + \frac{a}{D} + \rho\right) = \frac{a + \epsilon}{ID} + \frac{\rho}{I}$$

Since $N = \frac{\omega}{2\pi}$

$$N = 17 \text{ cycles/sec} \quad \text{for curve A'}$$

$$N = 18.3 \text{ cycles/sec} \quad \text{for curve B'}$$

These figures are in perfect agreement with the experimental curves A' and B'.

In conclusion it should be borne in mind that the inclination of the spindle axis, that is the angle of play, has only a small effect on the stability of the oscillations. Therefore this angle might just as well be determined from other considerations such as structural, geometrical, or operational characteristics.

B. Effect of Viscous Friction

Viscous friction is called a resistance proportional to the velocity of motion. Now suppose that the castering of the wheel is braked by a torque equal to $f = \frac{d\theta}{dt}$, that is

$$\frac{f}{a}\frac{dz}{dt}$$

The same line of reasoning employed in the foregoing for treating the case of restoring torque will produce the system

$$\frac{1}{v^2}\frac{d^2y}{dt^2} = -R\underline{M} \tag{16}$$

$$z - y = TF \tag{17}$$

$$\frac{I}{a}\frac{d^2z}{dt^2} = -F(a + \epsilon) - \underline{M} - \frac{f}{a}\frac{dz}{dt} \tag{18b}$$

$$\underline{M} \doteq \frac{1}{Sv}\frac{dy}{dt} + \frac{z}{Sa} - F\epsilon \tag{19}$$

The values of F and $\underline{M}$ from equations (17) and (19) entered in equations (16) and (18) yield the two equations

$$\frac{1}{Rv^2}\frac{d^2y}{dt^2} + \frac{1}{Sv}\frac{dy}{dt} + \frac{\epsilon}{T}y + \left(\frac{1}{Sa} - \frac{\epsilon}{T}\right)z = 0 \tag{20}$$

$$\frac{1}{Sv}\frac{dy}{dt} - \frac{a}{T}y + \frac{I}{a}\frac{d^2z}{dt^2} + \frac{f}{a}\frac{dz}{dt} + \left(\frac{1}{Sa} + \frac{a}{T}\right)z = 0 \tag{21b}$$

The values of z from equation (20) entered in equation (21b) give the differential equation

$$\frac{I}{Rav^2}\frac{d^4y}{dt^4} + \left(\frac{I}{Sav} + \frac{f}{Rav^2}\right)\frac{d^3y}{dt^3} + \left(\frac{I\epsilon}{Ta} + \frac{1}{SRav^2} + \frac{a}{TRv^2} + \frac{f}{Sav}\right)\frac{d^2y}{dt^2} +$$

$$\left(\frac{a + \epsilon}{TSv} + \frac{f\epsilon}{aT}\right)\frac{dy}{dt} + \frac{a + \epsilon}{TSa}y = 0 \tag{39}$$

It is readily seen that the second remark made following equation (22) is applicable here too. To obtain the differential equation of motion of the wheel, simply replace y by z in the preceding equation. The result is a linear differential equation of the 4th order of the form

$$a_0z^{IV} + a_1z^{III} + a_2z^{II} + a_3z^{I} + a_4z = 0$$

In connection with equation (22), it was seen that Routh's conditions for such a stable oscillation are

$$a_1 a_2 > a_0 a_3$$

$$a_1 a_2 a_3 > a_0 a_3^2 + a_1^2 a_4$$

The direct study of these equations being too laborious, will be abandoned in favor of the simplified theory. It is evident that this theory produces the equation

$$\frac{I}{Sav}\frac{d^3y}{dt^3} + \left(\frac{I\epsilon}{Ta} + \frac{f}{Sav}\right)\frac{d^2y}{dt^2} + \left(\frac{a + \epsilon}{TSv} + \frac{f\epsilon}{aT}\right)\frac{dy}{dt} + \frac{a + \epsilon}{TSa} y = 0 \tag{40}$$

obtained by disregarding all terms having R in the denominator of equation (39).

Incidentally, this is the same equation obtained by disregarding all the terms of higher degree in $1/v$, in equation (39), that is the terms with $1/v^2$.

The motion defined by the simplified theory can, therefore, still be regarded as the asymptotic limit of motion defined by the general equation when the velocity increases indefinitely. This remark applies also to equation (35) obtained in the theory of the restoring torque from equation (33), and to equation (6) with respect to equation (22).

Equation (40) is of the form

$$a_0 y^{III} + a_1 y^{II} + a_2 y^{I} + a_3 y = 0$$

and all its coefficients are, of necessity, positive. The unique condition of Routh's stability is then

$$a_1 a_2 \geqslant a_0 a_3$$

that is, by taking the fundamental relationship $S\epsilon = D$ into account,

$$\left(\frac{ID}{T} + \frac{f}{v}\right) \times \left(\frac{a + \epsilon}{v} + \frac{fD}{a}\right) \geqslant \frac{I(a + \epsilon)}{av}$$

which can also be written

$$ID^2fv^2 + \left[DTf^2 + I(a + \epsilon)(aD - T)\right]v + Ta(a + \epsilon)f \geqslant 0 \qquad (41)$$

In relation to the velocity v the left hand side represents a trinomial of the 2nd order admitting of two real roots of the same sign, v_1 and v_2, or else imaginary. Thus the condition of stability is realized in the following two cases:

(1) If the discriminant Δ of the trinomial (41) is negative

(2) If the discriminant Δ is positive and if, in addition, $v < v_1$ or $v_2 < v$

This argument assumes that v can take any algebraic value. But v is necessarily positive, which also restricts the scope of the conclusions, as will be seen. The discriminant Δ is written

$$\Delta = D^2T^2f^4 - 2IDT(a + \epsilon)(aD + T)f^2 + I^2(a + \epsilon)(aD - T)^2$$

With respect to f^2, Δ is a trinomial of the second degree having two positive roots

$$f_1 = \sqrt{\frac{I(a + \epsilon)}{DT}}\left(\sqrt{T} - \sqrt{aD}\right)$$

$$f_2 = \sqrt{\frac{I(a + \epsilon)}{DT}}\left(\sqrt{T} + \sqrt{aD}\right) \qquad (42)$$

Before proceeding to the discussion, three preliminary remarks are noted:

(1) If $aD > T$, the left side of the inequality (41) is a sum of positive terms and the stability condition is confirmed.

(2) If $DTf^2 + I(a + \epsilon)(aD - T) > 0$, that is if $f > \varphi$ by substituting

$$\varphi = \sqrt{\frac{I(a + \epsilon)}{DT}} \sqrt{T - aD}$$

the stability condition is proved again for the same reason.

(3) When $aD < T$, then $f_1 < \varphi < f_2$.

The second of this double inequality is evident. As to the first, it is sufficient to show that

$$aD + T - 2\sqrt{aDT} < T - aD$$

that is

$$2aD < 2\sqrt{aDT}$$

or else

$$aD < T$$

which is true by assumption.

So, if $aD < T$ and $f_1 < f$ the oscillation is always stable whatever the velocity may be, because

(a) either $f_1 < f < f_2$ and $\Delta < 0$

(b) or $f > \varphi$

When $f < f_1$, then $\Delta > 0$ and the oscillation is stable only when $v < v_1$, or $v_2 < v$.

To sum up: stability exists no matter what the velocity may be

(1) if $aD > T$

(2) if $aD < T$ and $f > f_1$

But, when $aD < T$ and $f < f_1$, the oscillations are unstable for any velocities between v_1 and v_2.

According to these conclusions when a tire shimmies, the phenomenon is, as a rule, observed only in a certain speed range (v_1, v_2), which can become quite extended.

To overcome shimmy, if aD is not $>T$, it is sufficient to use a hydraulic damper having a coefficient of viscous friction f greater than the coefficient of friction f_1 defined by equation (42). This solution has been frequently used, especially by the Americans. Unfortunately, this solution is not without drawbacks, because it is not simple, it increases the cost, requires a certain amount of maintenance, and increases the weight of the vehicle, which is particularly annoying when an airplane is involved. The best solution is very likely the simple realization of the relation $aD > T$.

The coefficient of minimum viscous friction to overcome shimmy is therefore f_1. For a value of f slightly below f_1, shimmy still exists for velocities near to v_0 given by the equation

$$v_0^2 = \frac{Ta(a + \epsilon)}{ID^2} \tag{43}$$

obtained by substituting $f = f_1$ in equation (41). This velocity v_0 can therefore be regarded as the most dangerous velocity for shimmy. With respect to the velocity v_m defined by equation (12a), that is relative to the maximum inversion velocity in the absence of the damper, we get therefore

$$\frac{v_0}{v_m} = \frac{2\sqrt{aDT}}{aD + T} \tag{44}$$

It follows, as is readily seen, that, as a rule

$$v_0 \leqslant v_m$$

C. Effect of Solid Friction

Solid friction or, as it is also called, constant friction, is a constant resistance independent of the speed of motion. Suppose that the castering of the wheel is braked by a constant torque $\underline{C}$. The case of solid friction is more difficult to treat analytically than that of viscous friction, but if it is only a question of defining the torque of solid friction $\underline{C}$ sufficient to prevent shimmy, the following expedient can be resorted to: the solid or the viscous friction damper, both act through the dissipation of energy transmitted to the wheel by the reactions of the ground. It therefore seems logical to assume that the solid friction damper will be sufficient to eliminate shimmy when it absorbs at each oscillation the same amount of energy as the viscous friction damper, that is with f_1 (formula (42)) for coefficient of viscous friction.

The energy dissipated during a quarter cycle of amplitude θ_m by solid friction damper is $W = \underline{C}\theta_m$. On the other hand, the energy dissipated by a viscous friction damper under the same conditions is

$$W = \int_0^{\theta_m} f \frac{d\theta}{dt} d\theta$$

When this damper is exactly able to make this oscillation represent the boundary between stability and instability, a sinusoidal oscillation

$$\theta = \theta_m \sin \omega t \quad \text{and} \quad f = f_1$$

is produced.

The preceding integral gives

$$W = \frac{\pi}{4} f_1 \omega \theta_m^{\ 2}$$

Equating the two values of W produces

$$\underline{C} = \frac{\pi}{4} f_1 \omega \theta_m \tag{45}$$

In this formula all parameters, save the angular frequency ω, are known. It was shown that ω is an increasing function with the velocity but varying little with this velocity. So, when v increases indefinitely the limiting value of ω should be taken. This value is, as seen,

$$\omega = \sqrt{\frac{a + \epsilon}{ID}}$$

and formula (45) becomes thus

$$\underline{C} = \frac{\pi}{4} \frac{a + \epsilon}{D} \frac{\sqrt{T} - \sqrt{aD}}{\sqrt{T}} \theta_m \tag{46}$$

Hence the expression of the solid friction torque is sufficient and even a little more than sufficient to stifle shimmy. This torque is proportional to the maximum amplitude θ_m while the viscous friction coefficient f_1 necessary to achieve the same purpose is independent of θ_m. This remark is extremely important, because it explains the fact pointed out frequently that, when solid friction is involved, shimmy is not produced in general in the absence of an initial displacement of sufficient amplitude. In other words, when solid friction is involved - and there always is more or less friction in the spindle axis, even in the absence of a special damper - shimmy, in order to be produced, has to be initiated or stimulated.

To avoid such initiation at landing, it might be useful to employ a wheel castering mechanism to keep the angle small at the moment of contact with the ground. So in formula (46) for the practical calculation of torque $\underline{C}$, one should use for θ_m the maximum amplitude likely to be accidently produced. The determination of this angle θ_m raises some difficulties. It has already been proposed to take for θ_m the angle at which the skidding of the tire on the ground occurs. This is justified if one assumes that for the greater angles sufficient braking action is assured by the skidding of the tire on the ground. Under these conditions formula (46) can be given a different form.

Consider a sinusoidal oscillation

$$\theta = \theta_m \sin \omega t$$

and assume, as in the approximate theory, that the oscillation of the wheel is largely due to the lateral force F exerted at a distance ϵ behind the point of contact. In this case

$$z = a\theta_m \sin \omega t \qquad \frac{d^2z}{dt^2} = -a\theta_m\omega^2 \sin \omega t$$

$$\frac{I}{a} \frac{d^2z}{dt^2} = -F(a + \epsilon) \quad \text{hence} \quad F(a + \epsilon) = I\theta_m\omega^2 \sin \omega t$$

and consequently

$$\theta_m = F_m \frac{a + \epsilon}{I\omega^2}$$

F_m being the lateral force which produces the skidding. On assuming that ω^2 is still approximately equal to

$$\frac{a + \epsilon}{ID}$$

formula (46) becomes

$$\underline{C} = \frac{\pi(a + \epsilon)}{4} \frac{\sqrt{T} - \sqrt{aD}}{\sqrt{T}} F_m \tag{47}$$

The limitation of the amplitude of shimmy as a result of the skidding of the tire when the lateral force F reaches the limit of adhesion F_m, explains in particular why the terrain most favorable to intense shimmy is macadam and dry and rough concrete, and why on the other hand, gravel and light soil have a certain damping effect on the oscillations of shimmy and thus limit the amplitude. It also explains why the load is a factor having considerable effect on the intensity of shimmy. All these facts have been checked many times on automobile wheels as will be shown later.

Many designers have adopted the solid friction damper as a means to stop shimmy. This solution has the advantage of being simpler and cheaper than the hydraulic damper, but it has the drawback of "stiffening" the castering of the wheel and so making it harder to maneuver the airplane on the ground. This drawback makes this solution to the problem extremely annoying in the case of airplanes of large weight, and practically limits its use to light aircraft. Furthermore, it is very difficult to ensure a well defined and perfectly constant value for solid friction. There always is the risk of too much or not enough tightness. All these drawbacks show that solid friction is not the ideal remedy against shimmy.

Tests made during the war in Germany by the Motor Institute, a branch of the Institute of Technology at Stuttgart, and by the Consolidated Vultee Aircraft Corporation in the U. S. A. (See Aero Digest of March 15, 1944, pp. 134 to 137; Conquest of Nose Wheel Shimmy by C. B. Livers and J. B. Hurt) have shown that nose wheel shimmy of the tricycle landing gear could be eliminated by the use of two wheels side by side on one axle. This remedy is similar to the preceding one, because the two wheels are clamped on the same axle with the possibility of differential rotation. The castering of the unit is, of necessity, accompanied by a certain skidding of the wheels on the ground, skidding equivalent to a solid friction. This solution presents the same drawbacks as the foregoing, it makes handling on the ground difficult, especially when a heavy airplane is involved.

Considerations of practical hydraulic dampers.

In the aforementioned study of viscous friction it had been assumed that the resistance was rigorously proportional to the rate of motion. In practice this law of damping is never realized for actual hydraulic dampers, because the true law of resistance in terms of velocity is the resultant of three effects:

(1) An effect of the viscosity of the liquid corresponding somewhat to a resistance proportional to the velocity.

(2) A kinetic effect or so-called hydraulic friction or turbulence, corresponding to a resistance proportional to the square of the velocity. This is a resistance of purely kinetic origin that is produced in every device forcing a liquid through a narrow orifice, even if a liquid without appreciable viscosity is involved.

(3) A valve effect, the preceding orifices being often duplicates of spring valves that remain closed as long as the pressure remains below a certain value, depending on the force of the spring. On opening they increase the area of the passage and prevent the torque from increasing with the square of the speed. All sorts of dampers with varied characteristics are obtained by appropriate control of the two compensating effects.

As a rule, a damper is characterized by the curve representing the resistance it offers to motion in terms of the rate of the motion. The ordinary dampers have generally characteristic curves similar to that shown in figure 36. The dashed lines represent the characteristic curve of a solid friction damper and the theoretical curves of pure viscous and pure hydraulic friction. The viscosity of a liquid decreases when the temperature increases, so that the typical curve of an actual damper is always deformed through diminution of all its ordinates. With such a graphical representation it can be immediately determined whether or not shimmy disappears if one assumes that two dampers are equivalent from the point of view of oscillation stability when they absorb the same amount of energy during one period. In that case, the viscous damping indicated by the theory (formula(42)) is simply transferred on to the characteristic diagram of the damper which gives a straight line passing through the origin. Considering the shape of the curve given for an actual damper it is seen that shimmy is likely to occur at low angular velocities and hence at low amplitudes. If the curve of the viscous damping required is tangent to the damper characteristic at zero velocity, the damper is exactly strong enough to prevent divergent oscillations. If the slope is less than that of the damper characteristic at zero velocity, it is more than sufficient to eliminate shimmy. Conversely, if the slope is greater, the torque at low velocities should be greater than that that can be given by the damper, and shimmy results. Lastly, if curve of the damping necessary intersects the curve of the damper at a comparatively high velocity above which the damper gives a more than ample torque, shimmy amplitude is limited to that at which the energy received by the wheel at each cycle is equal to the energy dissipated by the damper. If the curve of the resistance is definitely above the characteristic curve of the damper, the latter is powerless to limit the amplitude of shimmy. As the smallest amplitudes possible are already negligible, the only solution is to stifle shimmy completely. On the other hand, the theory of shimmy being applicable only to low amplitudes, if the damping provided is insufficient for stifling shimmy completely, the amplitude attained may be high enough so that the equations are no longer applicable, and the motion continues then to increase in intensity instead of remaining in a certain state of equilibrium. This is the reason why, in certain cases, it was deemed preferable to adopt a greater damping than strictly necessary in order to assure convergence; any oscillatory effects due to deviations from linearity that could occur are thus damped out.

Equation (47) presents an unusual feature. Taking the coefficients which we computed for the Schlippe-Dietrich tire, that is

$\epsilon = 4$ cm $\qquad T = 157 \times 10^{-10}$ (C.G.S. system) $\qquad D = 15.7 \times 10^{-10}$ (C.G.S. system)

and computing by this equation the variations of the friction coefficient $\underline{C}$ necessary to stifle shimmy for various caster lengths a, the following results are obtained:

a = 0 cm	$\underline{C} = 3.14\ F_m$ C.G.S. system	a = 5	$\underline{C} = 2.09\ F_m$ C.G.S. system
0.5	2.75	6	1.79
1	2.69	7	1.44
2	2.62	8	1.04
3	2.5	9	.56
4	2.32	10	0

The data are represented graphically in figure 37. Although this curve decreases consistently when a varies from 0 to $\frac{T}{D} = 10$, it is seen that $\underline{C}$ still is comparatively great for a values of the order of magnitude of 6 cm, the decrease of $\underline{C}$ being especially rapid when a approaches T/D. But since $\underline{C}$ measures to some extent, the eventual intensity or the danger of shimmy, it follows that the usual caster lengths are displaced precisely in the zone particularly dangerous for shimmy. The matter becomes plainer if one notes that on certain tires the curve $\underline{C}$ passes through a maximum for a value of a varying between 0 and T/D. In fact if it is assumed that $\epsilon = 2$ instead of $\epsilon = 4$ for the preceding tire, a figure still likely for a tire of that size, equation (47) then permits the calculation of the following values:

a = 0 cm	$\underline{C}$ = 1.57 F_m C.G.S. system	a = 5 cm	$\underline{C}$ = 1.63 F_m C.G.S. system
0.5	1.52	6	1.44
1	1.61	7	1.18
2	1.74	8	.865
3	1.77	9	.475
4	1.74	10	0

The results are represented by the curve of figure 38. The curve $\underline{C}$ passes through a maximum near a = 3. This remark can be illustrated by an analytical calculation: computing the derivative of $\underline{C}$ from equation (47) gives

$$\frac{d\underline{C}}{d\overline{a}} = \frac{\pi}{8}\sqrt{\frac{D}{aT}}\left(3a - 2\sqrt{\frac{T}{D}}\sqrt{a} + \epsilon\right)$$

when a is infinitely small the foregoing quantity assumes a negative infinite value which makes it possible to define the shape of the curve $\underline{C}$ for very small values of a. When $\frac{T}{D} > 3\epsilon$, the bracketed term (a trinomial of the second degree with respect to $\sqrt{a}$) cancels out for the two roots

$$\sqrt{a} = \frac{\sqrt{\frac{T}{D}} \pm \sqrt{\frac{T}{D} - 3\epsilon}}{3}$$

It is readily apparent that the smallest of these values of a corresponds to a minimum of $\underline{C}$ and the largest to a maximum. In the present case this calculation gives

$a_1 = 0.15$ cm $\quad\quad$ $\underline{C}_{min} = 1.48\ F_m$ C.G.S. system

$a_2 = 2.96$ cm $\quad\quad$ $\underline{C}_{max} = 1.77\ F_m$ C.G.S. system

The condition necessary and sufficient for curve $\underline{C}$ to assume a maximum is

$$\frac{T}{D} > 3\epsilon$$

This condition is realized on certain tires. The usual caster lengths are then precisely the most dangerous for shimmy. This remark stresses the importance of increasing the caster length to combat shimmy at its very source. Equation (12) makes it possible to establish that, when the velocity v varies, the maximum divergence λ_m is given by the equation

$$16DT^2I\lambda_m^{\ 2} = (a + \epsilon)(aD - T)^2$$

But a calculation readily proves that when $\frac{T}{D} > 2\epsilon$ (a condition realized for a large number of tires) and a is made to vary, this divergence λ_m passes through a maximum

$$\Lambda = \frac{T + D\epsilon}{6DT} \frac{T + D\epsilon}{3I}$$

for a caster length

$$a = \frac{T - 2D\epsilon}{3D}$$

The coefficient $\underline{C}$ seems better than the divergence λ for appraising the danger of shimmy.

Doctor Langguth's Experiments (1941).

It is timely to compare those theoretical conclusions with the experimental results obtained in 1941 by Doctor Langguth and reported by M. P. Mercier. He dealt with experiments on 260 × 85 tires, mounted on a rolling mat, at a speed of 50 km/h. The author recorded the maximum angular swivel or deflections of the wheel in terms of caster lengths for castering angles of -3°, 0°, and $+3^\circ$. Figure 39 summarizes the results. The four essential conclusions of this work are the following:

(1) The caster angle is of little importance

(2) The angular deflections pass through a maximum at $a = 7$ cm approximately

(3) They cancel out at around $a = 12$ cm

(4) Increasing the load increases the angular deflections practically in the same proportions

Before interpreting these results, it is advisable to understand the meaning of angular deflection (swivel) studied by the German author, since this quantity does not figure directly in the present theory. Nevertheless the theory, considered superficially, seems to indicate that, every time there is an oscillatory divergence, the angular deflection should increase indefinitely. This apparent contradiction is due to the fact that, in order to treat the problem, the data must be linearized, which is, perfectly legitimate so long as the angular deflections attain no unduly great values. Furthermore, the present theory assumes the adhesion of the tire to the ground to be infinite, while in reality this adherence is limited. The limit of adherence explains in particular why the maximum angular deflections were approximately proportional to the load on the tire.

Undoubtedly the present theory does not permit the ordinates of the curves of Langguth to be tied quantitatively to the characteristic parameters of the tire, but it does seem justified to assume that, everything else being equal, the maximum angle must vary as $\underline{C}$, and the existence of the maximum discovered by Langguth seems to be in good agreement with the foregoing theoretical conclusions. But from the present writer's point of view the major importance of these experiments is that they establish the existence of a caster length of about 12 cm below which (for $a < 12$) the oscillations are unstable and above which (for $a > 12$) the oscillations remain stable. In other words, Langguth's experiments confirm quite well the essential conclusion of the present theory, that is the stability of the oscillations with a sufficiently large caster length. And this conclusion is quantitative as well as qualitative: Langguth's 260 × 85 tire was of the same size as that used by Schlippe and Dietrich. Therefore, it is legitimate to assume that these tires have practically the same characteristics T and D. But, it has been shown that the caster length $a = \frac{T}{D}$ of the Schlippe - Dietrich tire, necessary for stable oscillations was about 10 cm. The quasi-agreement of this figure with Langguth's experimental value is extremely remarkable, in as much as the slight residual discrepancy between the two values can be readily attributed to a difference in tire inflation in the two test series.

VI. MATHEMATICAL THEORY OF COMPLEX SHIMMY

The term complex shimmy describes shimmy with two degrees of freedom, the second degree of freedom resulting from a certain lateral elasticity of the pivoting axis.

As in the study of elementary shimmy, we shall start with the simplified theory, disregarding the torsional moment of the bottom part of the tire, that is assume that the drift directly accompanies the lateral force F which, in more exact terms, amounts to assuming the coefficient of turn R of the tire to be infinite.

To put the problem into mathematical form we return to figure 8 and to the line of reasoning that produced the system of equations (1), (2), and (3). But, first, in order to account for the lateral elasticity of the pivoting axis, it is advisable to replace figure 8 by figure 40:

With l as the lateral displacement of the pivot under the action of the lateral force F_0,

$$l = T_0 F_0$$

where T_0 is called the coefficient of the lateral elasticity of the pivot.

Equation (1) always gives the relation

$$\delta = \theta + \psi = DF$$

but now

$$\theta = \frac{z - l}{a}$$

so that equation (1) must be replaced by

$$\frac{1}{v}\frac{dy}{dt} + \frac{z}{a} - \frac{l}{a} = DF$$

Equation (2) is obviously not modified.

As to equation (3), it is transformed as follows: If I_0 is the inertia of the wheel about the vertical axis passing through the center of gravity of the wheel, then $I = I_0 + ma^2$, where m = mass of wheel. The angular acceleration of the wheel about this axis is a function of the sum of the moments of the forces F and F_0, hence

$$I_0 \frac{d^2\theta}{dt^2} = F_0 a - F\epsilon$$

now

$$\theta = \frac{z' - l}{a} \quad \text{and} \quad F_0 = \frac{l}{T_0}$$

Therefore

$$\frac{I_0}{a}\left(\frac{d^2 z}{dt^2} - \frac{d^2 l}{dt^2}\right) = \frac{al}{T_0} - F\epsilon$$

The introduction of a new variable l calls for a fourth equation, which is obtained by noting that the acceleration of the center of gravity O is due to the sum of the forces F and F_0. Thus

$$m \frac{d^2 z}{dt^2} = -F - F_0 = -F - \frac{l}{T_0}$$

Hence the following system

$$\frac{1}{v}\frac{dy}{dt} + \frac{z}{a} - \frac{l}{a} = DF \tag{48}$$

$$z - y = TF \tag{49}$$

$$\frac{I_0}{a}\left(\frac{d^2 z}{dt^2} - \frac{d^2 l}{dt^2}\right) = \frac{al}{T_0} - F\epsilon \tag{50}$$

$$m \frac{d^2z}{dt^2} = -F - \frac{l}{T_0} \tag{51}$$

To obtain the differential equation in z defining the motion of the wheel, simply eliminate F, l, and y from these four equations, as follows:

The four equations are linear and homogeneous. So, if

$$y = Ae^{st}$$

is a particular solution of the system, three particular solutions for z, l, and F of the form

$$z = Be^{st} \qquad l = Ce^{st} \qquad F = De^{st}$$

are obtained. Thus the equation in s of the foregoing system or the characteristic equation of the system can be obtained directly by replacing the unknowns y, z, l, and F by the values above and then eliminating A, B, C, and D from the four equations of the system by Cramer's method of determinants. Hence we obtain an equation in s in the form

$$\begin{array}{cccc} (y) & (z) & (l) & (F) \end{array}$$

$$\begin{vmatrix} \frac{1}{v} s & \frac{1}{a} & \frac{1}{a} & -D \\ -1 & 1 & 0 & -T \\ 0 & \frac{I_0}{a} s^2 & -\frac{I_0}{a} s^2 - \frac{a}{T_0} & \epsilon \\ 0 & ms^2 & \frac{1}{T_0} & 1 \end{vmatrix} = 0$$

The development of the determinant gives the characteristic equation

$$\frac{mI_0}{v} s^5 + \frac{mI_0 D}{T} s^4 + \frac{1}{v}\left(\frac{I_0}{T} + \frac{I_0 + ma^2}{T_0}\right)s^3 + \frac{1}{T}\left[\frac{D\left(I_0 + ma^2\right)}{T_0} + m\epsilon\right] s^2 +$$

$$\frac{a(a + \epsilon)}{TT_0 v} s + \frac{a + \epsilon}{TT_0} = 0$$

The differential equation sought is thus

$$\frac{mI_0 TT_0}{v} \frac{d^5 z}{dt^5} + mI_0 DT_0 \frac{d^4 z}{dt^4} + \frac{I_0 T_0 + IT}{v} \frac{d^3 z}{dt^3} + (ID + m\epsilon T_0)\frac{d^2 z}{dt^2} +$$

$$\frac{a(a + \epsilon)}{v} \frac{dz}{dt} + (a + \epsilon)z = 0 \tag{52}$$

It should be noted that, when $T_0 = 0$, that is when the pivot has no lateral elasticity, the foregoing equation reverts to equation (6). Two particular cases with important conclusions will be analyzed.

1. Wheel Absolutely Rigid

Assume that simultaneously

$$T = 0 \qquad D = 0 \qquad \epsilon = 0$$

that is a wheel without any lateral elasticity, hence without drift. Equation (52) is simplified and becomes

$$\frac{I_0 T_0}{v} \frac{d^3 z}{dt^3} + \frac{a(a + \epsilon)}{v} \frac{dz}{dt} + (a + \epsilon)z = 0$$

It is readily seen that the condition of Routh's convergence can never be realized with such an equation. In such a case, if T_0 is not zero, that is if the pivot has a certain lateral elasticity, there always will be a divergent oscillation, that is shimmy.

The lateral elasticity of the pivot can be the sole generator of shimmy without it being necessary for the mechanical properties of the tire themselves being involved. For certain particular combinations of tire factors and speed, the lateral elasticity of the pivot can therefore become the factor facilitating shimmy.

2. Pivot With Great Lateral Freedom

Centering attention on parameter T_0 it is assumed that $T_0 = \infty$, that is that the pivot is absolutely free to shift laterally under the action of a force no matter how small. Equation (52) then becomes

$$\frac{mI_0T}{v} \frac{d^5z}{dt^5} + mI_0D \frac{d^4z}{dt^4} + \frac{I_0}{v} \frac{d^3z}{dt^3} + m\epsilon \frac{d^2z}{dt^2} = 0$$

Putting $\frac{d^2z}{dt^2} = u$, the preceding equation, which is independent of a is transformed into a differential equation of the 3rd order for which Routh's stability condition becomes, after simplifications,

$$I_0 > m \frac{T}{D} \epsilon$$

It is readily apparent that this condition is realized generally with ordinary tires. The ratio T/D is a length generally very close to the radius r of the tire; and considering that

$$I_0 = \frac{1}{K} mr^2$$

K is a coefficient generally near 2. The preceding inequality thus becomes $r > 2\epsilon$, which is amply confirmed for all customary tires.

Since the pivot is completely free to shift laterally, the oscillations are, naturally, convergent, that is there can be no shimmy. Hence the possibility of a new remedy for removing shimmy by giving the pivot complete lateral freedom.

Based upon more elementary reasoning of qualitative rather than quantitative nature, various American technicians had been led to assume that the possibility of a wheel to shift laterally without pivoting should be capable of suppressing shimmy. To realize this aim, two types of mechanical arrangements were recommended and tested, either on the airplane or on the belt-machine apparatus.

The first method consists in letting the free wheel slide on its axis, by allowing the necessary play at the end of the hub. The axis was slightly curved so that the wheel had a tendency to remain in the center. This method was tested by the NACA on the Weick W-1 airplane and revealed itself as being effective as long as the pivoting remained below 13° in one direction or the other. But other tests have given no satisfaction. In the light of the present theory it is easy to understand the reason for this failure. Whatever the perfection of lubrication, the friction of the wheel on its axle, in the motion of the lateral displacement, cannot be considered as negligible. This friction reestablishes, in a certain measure, a lateral restraint. It follows that, if the stability condition

$$I_0 > m \frac{T}{D} \epsilon$$

is not greatly exceeded, the phenomenon of shimmy may reappear.

The second method consists in utilizing two pivots coupled as illustrated in figure 41. The NACA tested this system in the laboratory and found that it eliminated shimmy. It was also studied by E. R. Warner (ref. 4). The operation is obvious. This arrangement realizes complete lateral freedom for the wheel pivot. It seems quite superior to the first, although structural complications are involved and landing gear retraction is made more difficult. The American authors, E. S. Jenkins and A. F. Donovan (ref. 5) believe, however, that it could be made as light as a damped system and that the elimination of the detrimental effects of damping friction on ground handling, as well as the elimination of the upkeep of a damper, would have appreciable advantages. The additional complications introduced by this system of double pivot are compensated for in part by dispensing with the scissors which transmit the shimmy torque to the damper in the ordinary arrangements.

Clearly, all the advantages presented by this arrangement are found in that advocated in the present report (simple lengthening of caster length) and which has the added advantage of greater simplicity.

The two foregoing examples prove that, in the fight against shimmy, the existence of a certain amount of lateral elasticity may sometimes turn out beneficial, sometimes harmful, depending on the magnitude of this elasticity and the values of the characteristic coefficients and the velocity. A clarification of this point calls for a complete study of the stability condition of equation (52). Unfortunately, too much paper work is involved to be undertaken by us. Furthermore there does not seem to be much practical interest in such a discussion; in fact, it has been shown that it is comparatively easy to stop shimmy in the case of a rigid pivot; and the realization of a lateral elasticity, properly speaking, in the pivot will present difficulties of construction and drawbacks of practical nature.

Effect of a restoring torque

Assume a restoring torque of the form $\underline{C}r = \rho\theta$. Obviously, equations (48), (49), and (51) remain unaltered. Equation (50) is replaced by

$$\frac{I_0}{a}\left(\frac{d^2z}{dt^2} - \frac{d^2l}{dt^2}\right) = \frac{al}{T_0} - F\epsilon - \frac{\rho}{a} z \tag{50a}$$

the same way as equation (18) was replaced by equation (18a) to obtain equation (33).

The result is a system of four equations which is resolved by means of the method used for equations (48), (49), (50), and (51). Consequently

$$\frac{mI_0TT_0}{v}\frac{d^5z}{dt^5} + mI_0DT_0\frac{d^4z}{dt^4} + \frac{IT + I_0T_0 + mTT_0\rho}{v}\frac{d^3z}{dt^3} + (ID + m\epsilon T_0 +$$

$$mDT_0\rho)\frac{d^2z}{dt^2} + \frac{a(a + \epsilon) + (T + T_0)\rho}{v}\frac{dz}{dt} + (a + \epsilon + D\rho)z = 0$$

If $\rho = 0$, equation (52) is obtained again.

Complete Theory of Complex Shimmy

If the torsional elasticity and the turning effect of the tire are involved, the foregoing line of reasoning results in a system of five equations.

$$\frac{1}{v^2}\frac{d^2y}{dt^2} = -R\underline{M}$$

$$z - y = TF$$

$$\frac{I_0}{a}\left(\frac{d^2z}{dt^2} - \frac{d^2l}{dt^2}\right) = -F\epsilon + \frac{al}{T_0} - \underline{M}$$

$$\underline{M} = \frac{1}{sv}\frac{dy}{dt} + \frac{z - l}{Sa} - F\epsilon$$

$$m\frac{d^2z}{dt^2} = -\left(F + \frac{l}{T_0}\right)$$

The differential equation sought is obtained by eliminating F, $\underline{M}$, l, and y from the five equations. This system can also be studied and resolved more rapidly by the method employed for equations (48), (49), (50), and (51). Even then the calculations are long and therefore not attempted.

VII. SHIMMY OF AUTOMOBILE WHEELS

Ever since the first automobiles started to appear in public certain cars revealed the existence of a very peculiar oscillatory phenomenon of the front wheels consisting of a periodic lateral wavering. This phenomenon generally occurred when the car reached or exceeded certain critical speeds. At first, it was called waddling or tacking. In the beginning, it seemed to present no particularly serious problem and car designers paid little attention to it. Moreover, at the slow speeds of cars in that era, it was successfully overcome by all sorts of empirical means, among which might be mentioned at random:

Toein of front wheels

Increasing the angle of camber

Use of tight shock absorbers

Stiff front springs

Stiff and irreversible steering

Higher tire pressure, etc.

All these attempts were no more than palliatives, the simple effect of which was to put the critical speed for the appearance of shimmy above the speed of utilization of the car. Shifting the critical speed was comparatively easy, except for racing cars where more energetic remedies were called for. The only truly effective remedy for this type of car was the use of very stiff suspensions and hard shock absorbers which meant the complete loss of comfort.

But this state of affairs became suddenly worse when the Americans tried to introduce the balloon tire. The much greater elasticity of this tire, so pleasant for greater comfort, actually, lowered the critical speed of oscillation of the front wheels considerably and this was aggravated as the suspension became more flexible and front wheel brakes came into wider use. On some cars, these phenomena were so violent that it became impossible to speed up even moderately. The oscillation was transmitted integrally to the axle and, because of gyroscopic interactions, was not just limited to a simple motion in a horizontal plane but also in the vertical plane, which accentuated and compounded the manifestations to an unusual degree. In some cases the axle seemed to do a fervent acrobatic dance, and this is why the Americans got to designate the whole of these phenomena by the name of an exotic dance then in vogue in the U. S. A. and the name shimmy has remained classical ever since.

For some years shimmy was regarded as a particularly grave ailment of balloon tires and even as a redhibitory vice to their commerical distribution. Nevertheless, the greater riding comfort on balloon tires tempted the engineers of every country to all sorts of ways to combat this parasitic and annoying phenomenon. But the technicians themselves did not agree on the origin and nature of shimmy, and even less on a rational method of avoiding its effects. The question reached such importance that a whole session of the U. C. Congress of Automobile Engineers in 1925, was devoted to it. Everybody stuck loyally to the results of his own observations and to his personal opinion. But the diversity of opinion of assumptions and remedies suggested, the utter lack of a general viewpoint on the nature of the subject, far from bringing any enlightenment, actually left the impression of profound inconsistency. It is only necessary to consult "La technique automobile et aerienne," vol. 16, no. 130, 3rd quarter 1925) which published the principal reports read at that session. It is particularly surprising to find that not one referred to the true origin of shimmy. The underlying cause of shimmy lies in the peculiar mechanical properties of the tire. It is a complex shimmy, the sizeable elasticity of the longitudinal suspension springs plays the part of the lateral elasticity of the pivot.

However, the complete mathematical theory of this shimmy of automobile wheels is more complicated than that of complex shimmy described in the foregoing and its external manifestations themselves are a little more dissimilar by reason of the following two factors:

(1) The reactions of the linkage or of the steering gear

(2) The gyroscopic reactions

The first idea of seeking the cause of shimmy in the mechanical properties of the tire is much earlier than the recent reports of American experimenters on airplane tires. On May 22, 1925 a French engineer, known for his many publications on the dynamics of the automobile and for his frequently original views, A. Broulhiet, gave a very interesting speech before the Société des Ingenieus Civile de France (see the report of this society (1925, pp. 540-554)) in which he stated that the drift of the tire was the true cause of shimmy. Perhaps Broulhiet's idea was not convincing enough, perhaps it was not accompanied by a sufficiently correct mathematical theory nor developed enough to have attention called to it as the only correct explanation of shimmy. In any case, it was not taken up by other theorists and did not receive the publicity it should have had. Incidentally, the development of automobile engineering has a rather unusual history and which proves that even a wrong theory can sometimes be very practical.

At this meeting of the Society of American Automobile Engineers in 1925, A. Healy, of the Dunlop Tire Company, presented a report in which he charged the gyroscopic effect of the wheels as being the cause of shimmy. The idea was taken up in France by an engineer of Russian extraction, Dimitri Sensaud De Lavaud, a specialist in automobile dynamics, who undertook to publish a complete mathematical theory of this gyroscopic effect in two installments (refs. 6 and 7).

It is true that such a study is of technical interest, because the gyroscopic reactions have an undesirable influence on the shimmying motions if the two front wheels are mounted on the same axle. But they can in no case be the true cause of shimmy because they are conservers of energy. Consequently the gyroscopic effect absolutely cannot explain the fundamental characteristic of shimmy which is the divergence or instability of oscillations resulting from a continuous transformation of kinetic energy of translation of the car into oscillating energy of the wheel.

This matter did not escape De Lavaud's attention, who explicitly remarked on it. To explain the divergence of the oscillations he attempted to bring in the rocking motion which naturally accompanies shimmying. To be sure, there is a real possibility of divergence but this scarcely perceptible rocking motion is so minute that its amount of energy per cycle is very small and surely too small to explain the intensity of shimmy especially when considering the many sources of damping which restrain the turning motion of the wheel.

In De Lavaud's theory this imperceptible rocking motion of the car is the sole possible cause of the divergence of oscillation. But it is plainly inadequate. The author was therefore compelled to regard shimmy not as a phenomenon of instability, that is divergence of oscillations, but as a simple resonance phenomenon arising from defective balancing or centering of the wheels. Today it is known as an absolute fact that shimmy is not a resonance phenomenon but due solely to the divergence of oscillations, which is entirely different. Even if it is true that the unbalance of the front wheels may facilitate and, in certain cases, actually cause artificial shimmy, experience proves that defective balancing of wheels or tires, whether automobile or airplane, is not the true cause of shimmy. No resonance phenomenon could ever explain the intensity so often observed. Furthermore, shimmy has been observed on perfectly balanced wheels rolling on level ground. In addition the phenomenon designated by the Americans with the name of kinematic shimmy, indicates clearly, that it can have an oscillatory motion without the least effect of inertia, which, however, is necessary in order to have resonance.

To sum up, the gyroscopic actions which accompany automobile wheel shimmy and which complicate the phenomenon can never be entirely the cause. This is readily apparent when considering airplane wheel shimmy studied under the name of elementary shimmy. In this case the gyroscopic actions are actually without any effect on the motion of the wheel, since their moment is then perpendicular to the pivoting axis which itself is absolutely rigid.

In short, De Lavaud mathematically defined the concept of the gyroscopic coupling of two oscillations, very important, it is true, but which did not evolve the true source of energy brought into play by shimmy. As a logical consequence of his theory, De Lavaud was forced to advocate the elimination of the gyroscopic effect by an appropriate mode of linkage of the front wheels to the chassis permitting the wheels only a vertical displacement in their own plane. Through this De Lavaud became the champion and forerunner of this type of suspension, which under the name of independent front wheel suspension achieved a certain amount of success for two reasons:

First, the independent front wheel eliminated, as foreseen, the gyroscopic reactions and their vitiating effects, not only in the case of shimmy but also for the passage of the wheel over any obstacle, and this advantage alone was enough to justify its use. But, and this is a curious historical fact, the independent front wheels also completely eliminated shimmy. In the face of these advantages one after the other automobile builder adopted this mode of suspension. So, with the problem of shimmy practically solved, the automobile engineers dropped its study.

Without any other justification it was accepted as an accomplished fact that the gyroscopic effects were the true cause of this phenomena, and up to this day the theory of De Lavaud is regarded as complete explanation. But now it has been proved that the gyroscopic actions are simply a phenomenon accompanying but not a primary cause of shimmy. How then can the fact that the independent wheels eliminate shimmy be explained?

Actually, all independent front wheel suspension systems have in common this peculiarity of totally suppressing the lateral elasticity of the front wheels, a lateral elasticity which was formerly considerable with the longitudinal springs generally used for suspending the chassis from the axle common to the two wheels. Owing to this fact, automobile wheels can develop only elementary shimmy with a restoring torque due to the elasticity of the steering gear. The law of motion then follows equation (33). To establish the stability conditions, reference is made to the discussion following this equation. In automobile wheels with a given caster length

$$aD < T$$

It follows that the unique condition of stability is $v < u$, that is

$$v^2 < \frac{a}{IR} + \frac{\rho T}{(a + \epsilon)IR}$$

To find the conditions of stability at any speed the car may reach simply consider a direction in which the steering is rigid or in other words, large enough ρ. This is precisely the combination realized on all modern cars.

VIII. CONCLUSIONS

The present report, instead of starting with long series of experiments and involving costly materials, begins with the discussion and exploitation of all previous experiments in enough detail to be practicable. Life is short and it serves no useful purpose to lose time in repeating the experiments already made with care and of which the results have been published. This method of work certainly saves time and money, because, taking into consideration that the comparison of the present theory with the many results obtained earlier by other experimenters, could leave no doubt about the physical value of this theory and it is this physical value which is the principal achievement of this work.

However, some experiments were carried out by the O.N.E.R.A. at Toulouse in order to support this study which involved an experimental check of the fundamental stability equation (23a).

$$(IRv^2 - a)(aD - T) > 0$$

This formula can be checked by ascertaining that the conditions of stability and those of shimmy in terms of caster length a and velocity v are those represented in figure 42. The experiments themselves merely involve checking the pressure or absence of shimmy, hence comparatively easy and inexpensive measurements, which are, in any case, less difficult and shorter than Kantrowitz's divergence measurements.

In all cases, the practical use of this theory requires the preliminary and systematic measurement of the characteristic factors T, D, S, R, and ϵ of the different tires used in aeronautics. It also is desirable to study the variations of these factors in terms of tire inflation and load. Such a study would make it possible to decide the form, size, and inflation pressure required to obtain the minimum for the quotient T/D, so as to be able to combat shimmy without having to resort to long caster lengths.

To conclude, it is a theory which seems to have already been supported by many experimental proofs and which is herewith presented to tire technicians and experimenters; I now let them speak.

Translated by J. Vanier
National Advisory Committee
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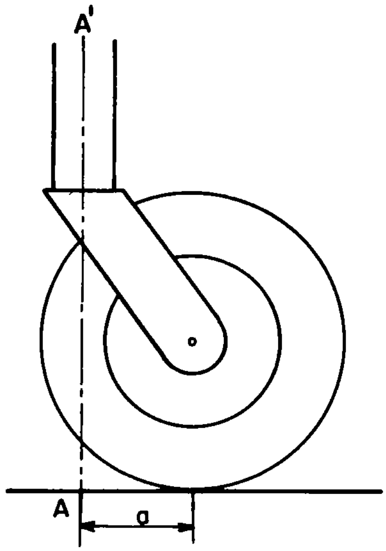


Figure 1

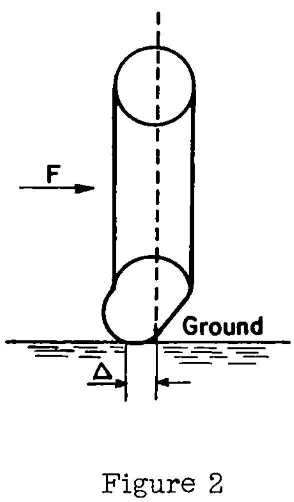


Figure 2

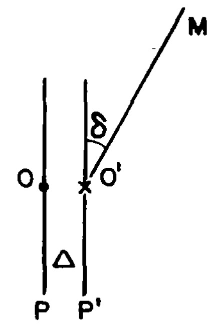


Figure 3

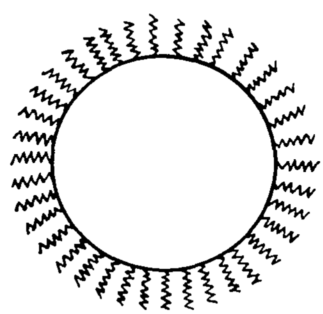


Figure 4

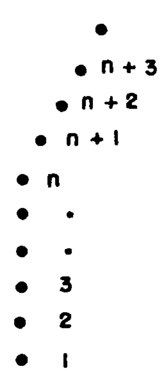


Figure 5

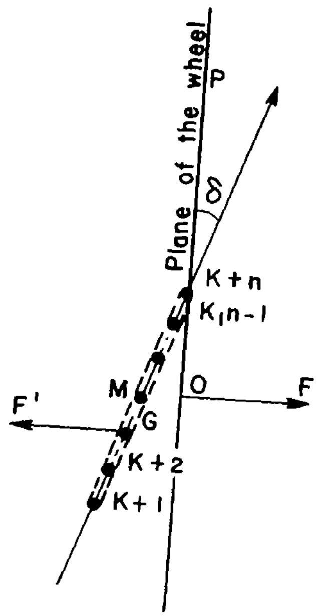


Figure 6

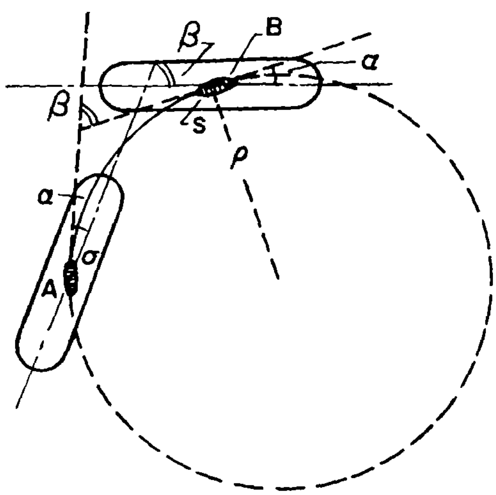


Figure 7

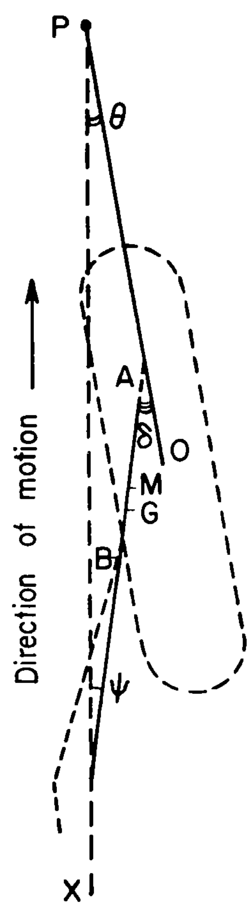
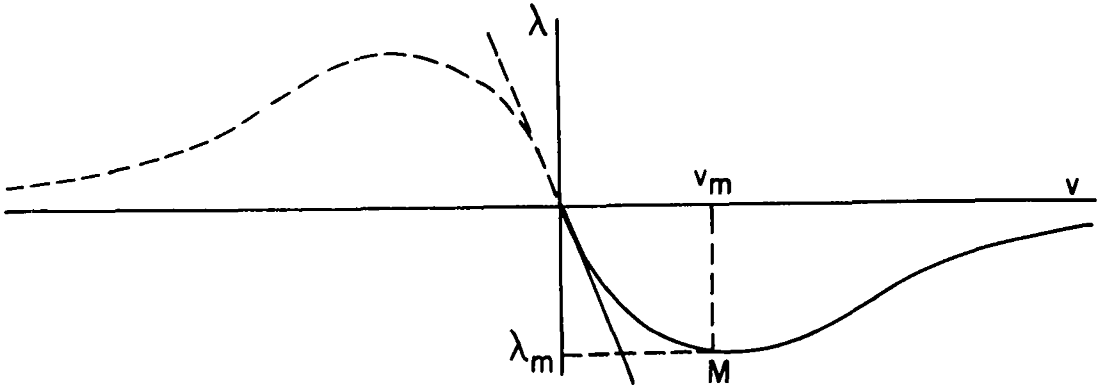


Figure 8



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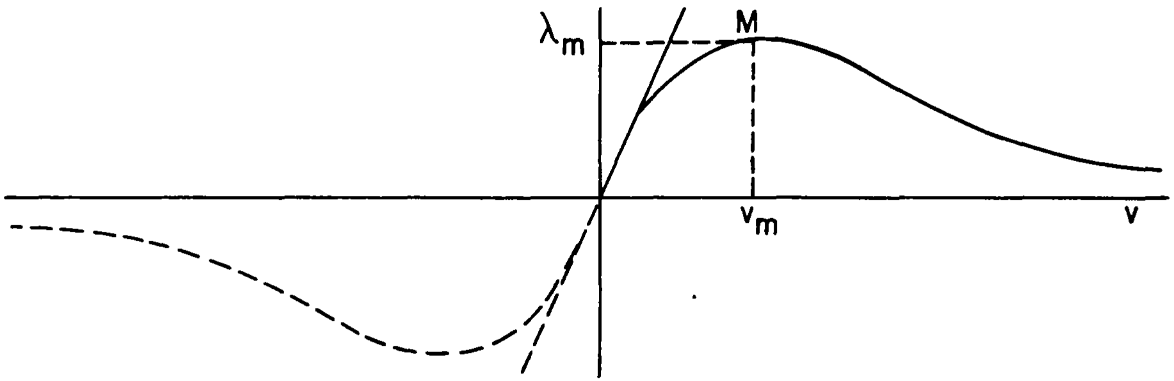

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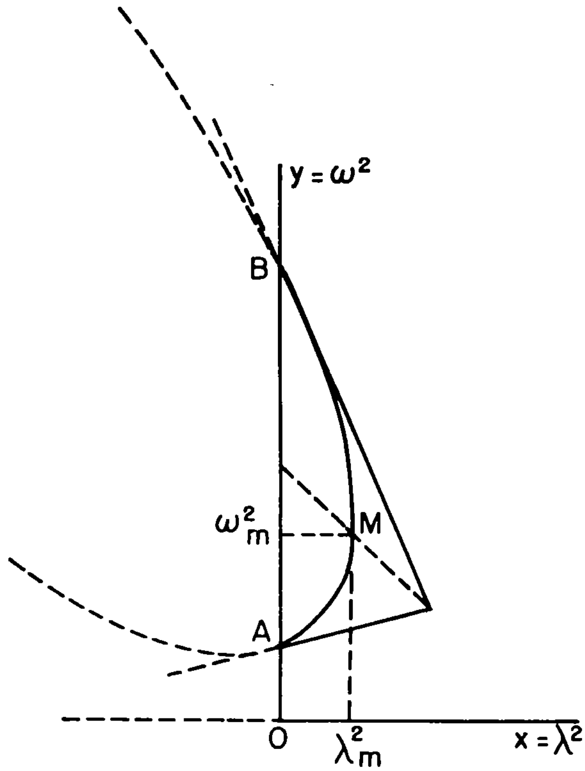

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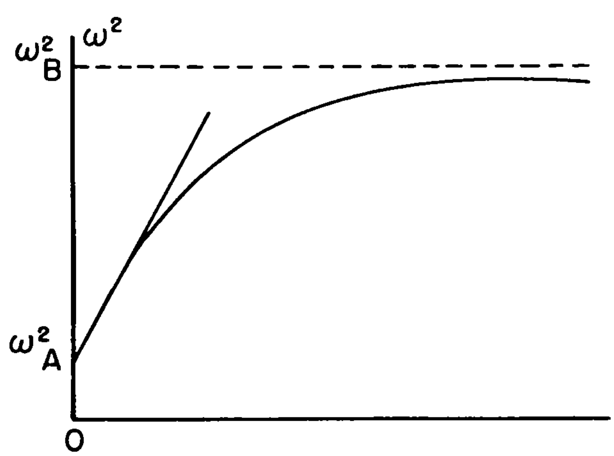

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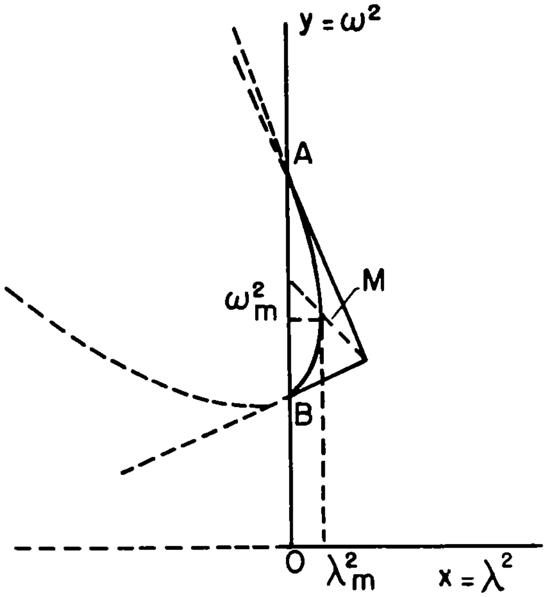

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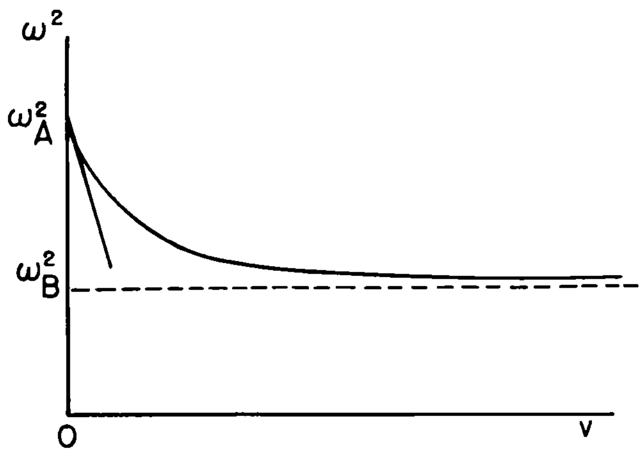

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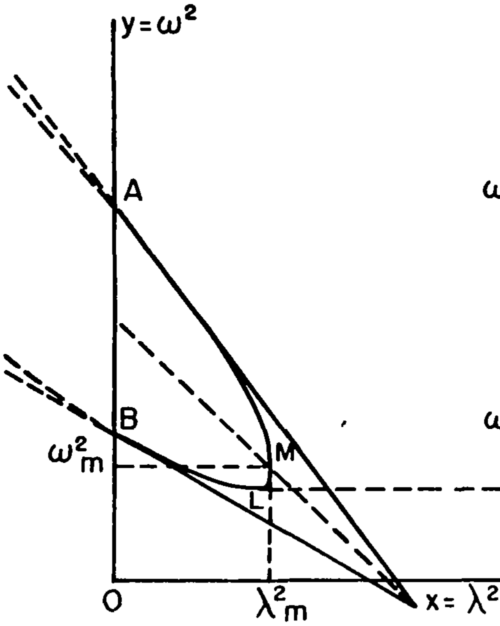

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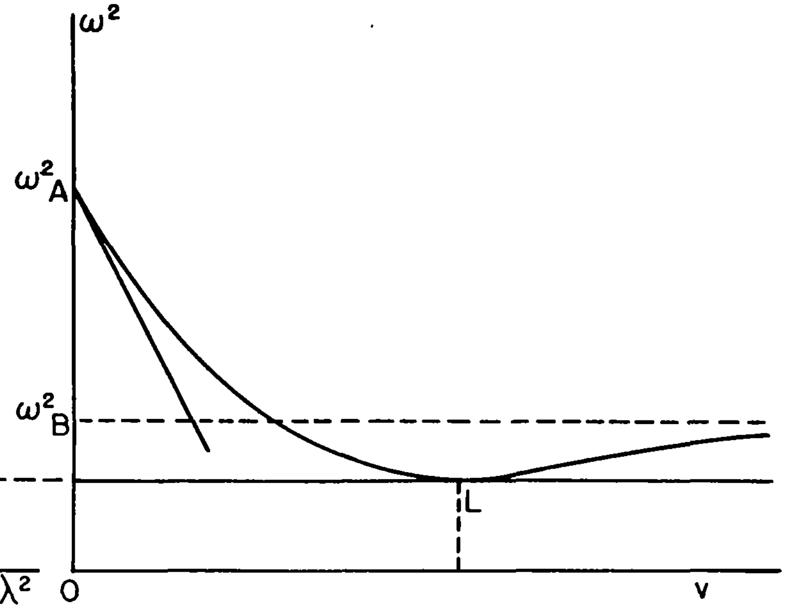

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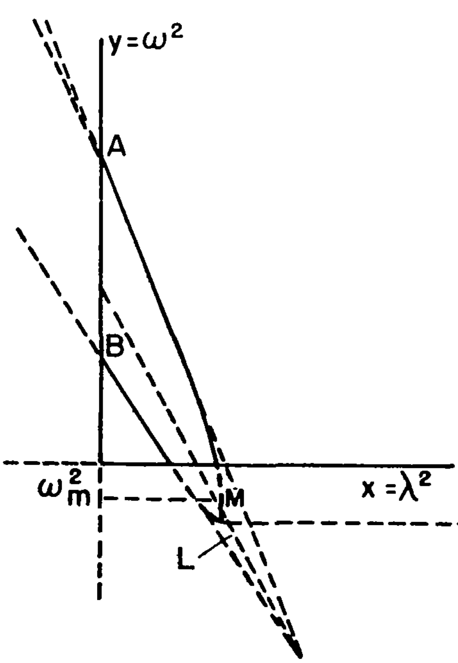

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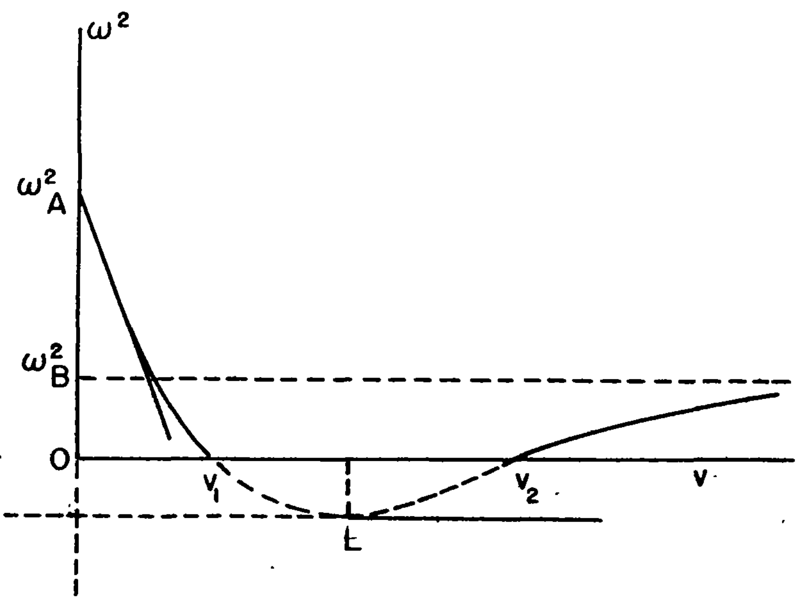

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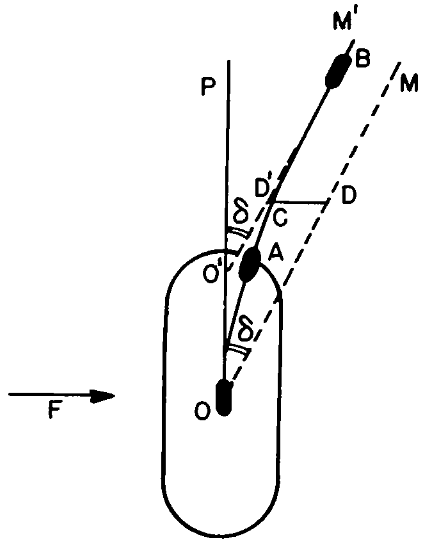

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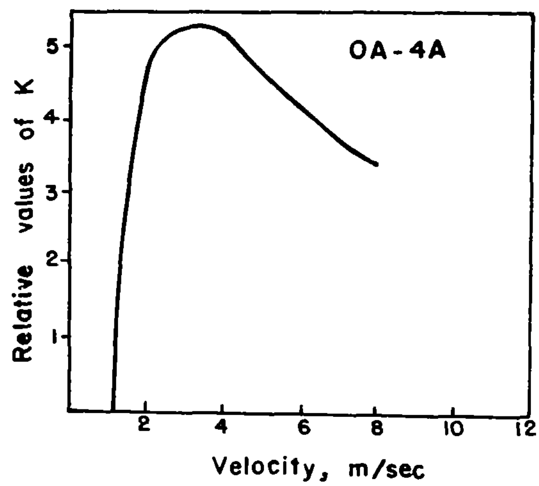

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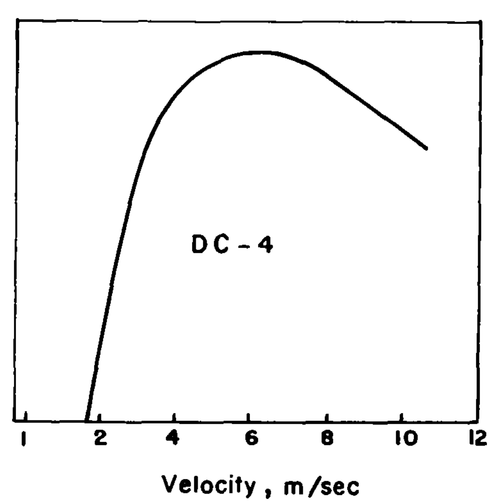

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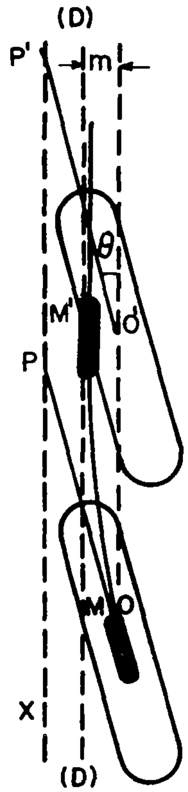

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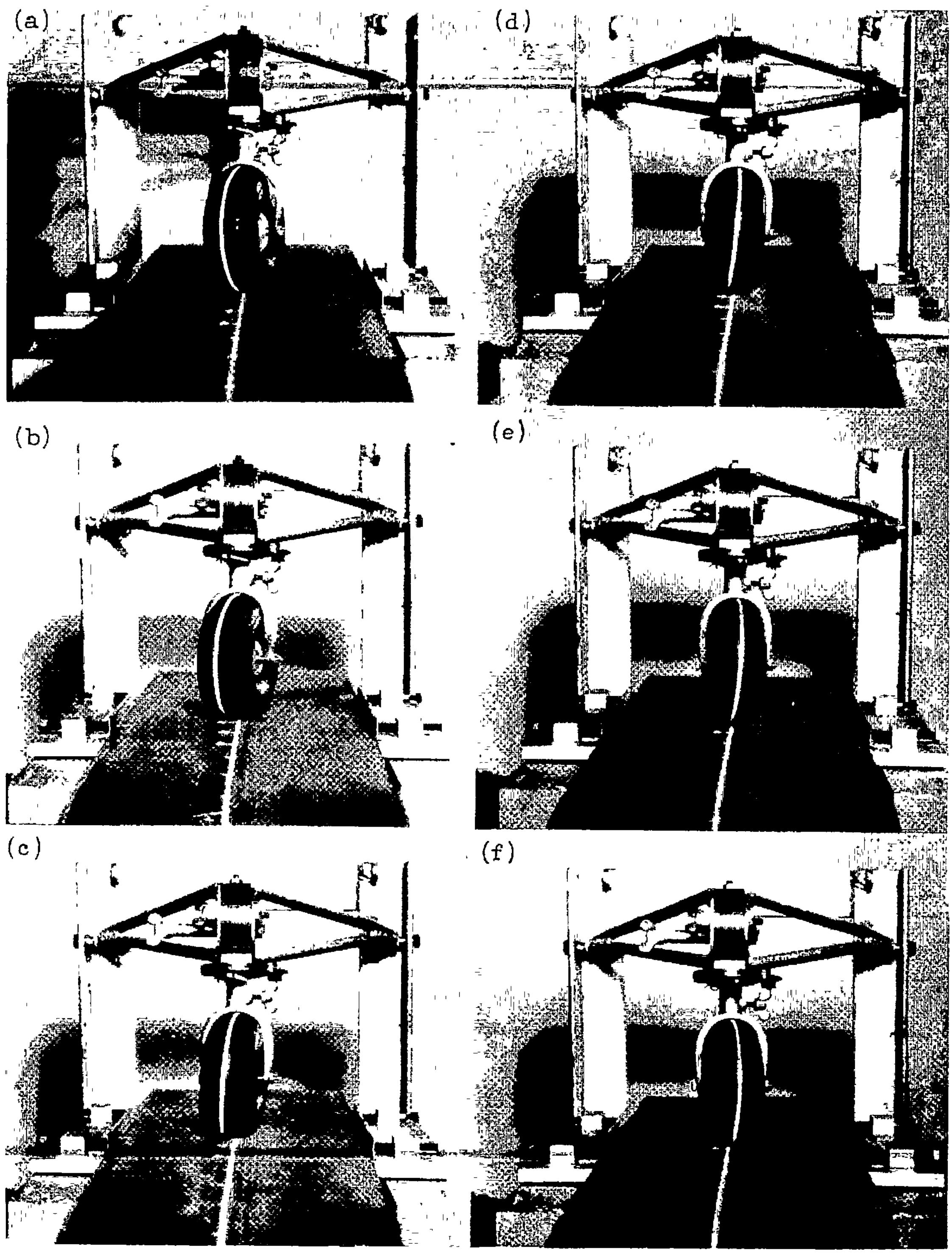

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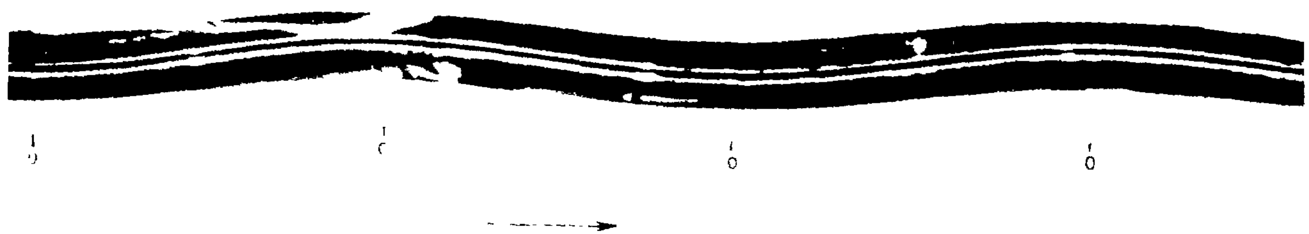


Figure 24.- Record of the track of a wheel executing a kinematic shimmy.

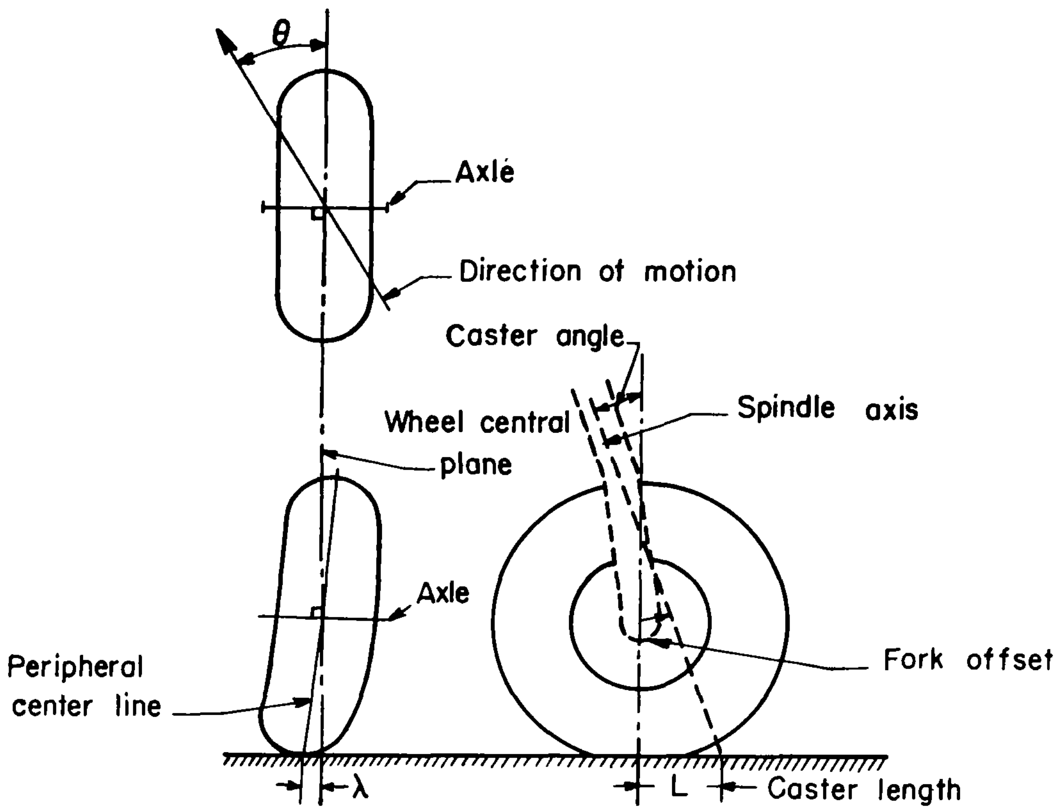


Figure 25.- Definition of forms. Illustrated directions positive.

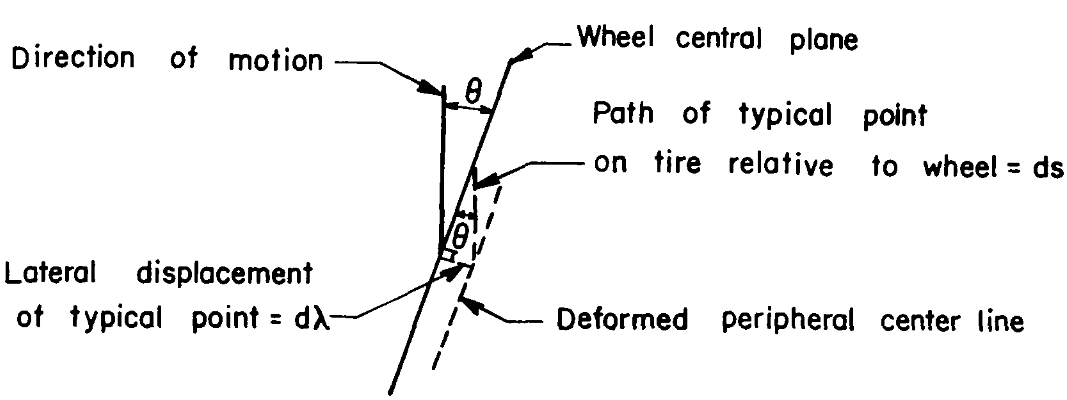


Figure 26.- Geometrical relations involved in kinematic shimmy, illustrating that $d\lambda = -\theta ds$.

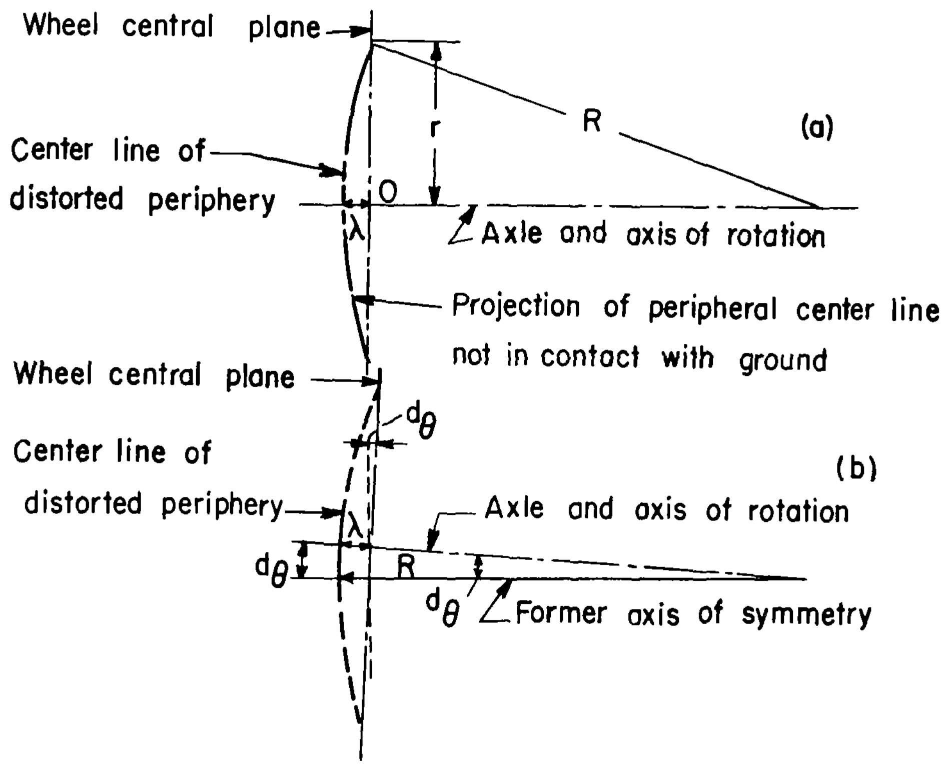


Figure 27.- Geometrical relations involved in kinematic shimmy illustrating that $d\theta = 2\lambda/r^2 = k_1\lambda$.

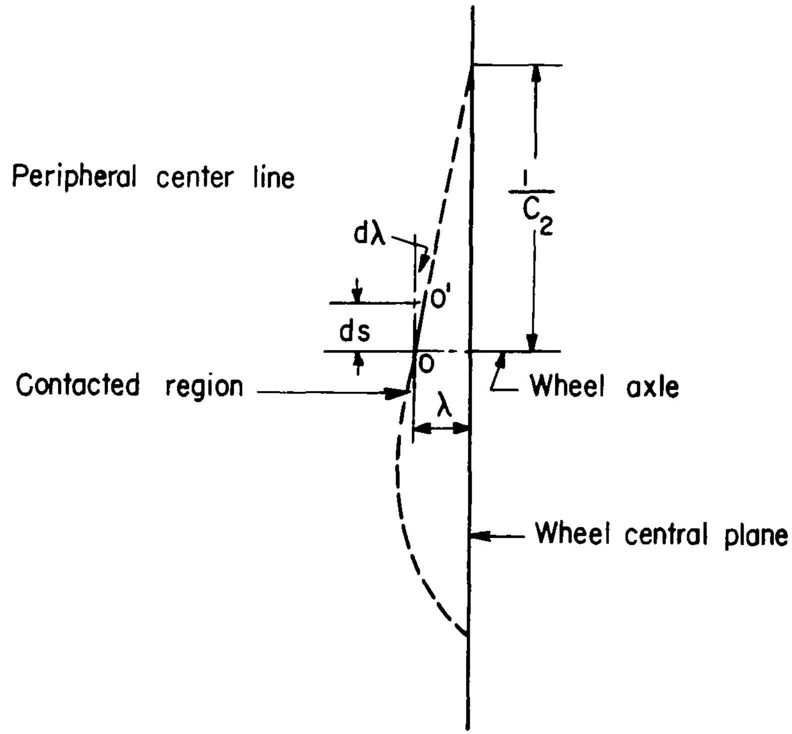


Figure 28.- Illustrating the contribution of asymmetrical strain to $d\lambda/ds$.

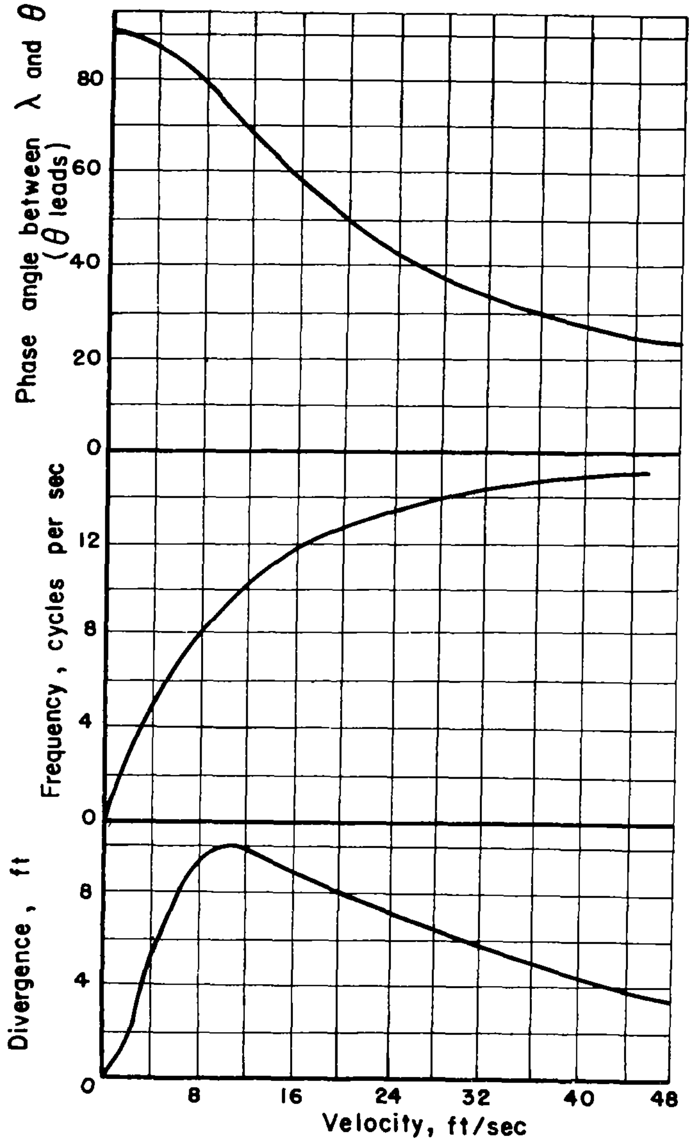
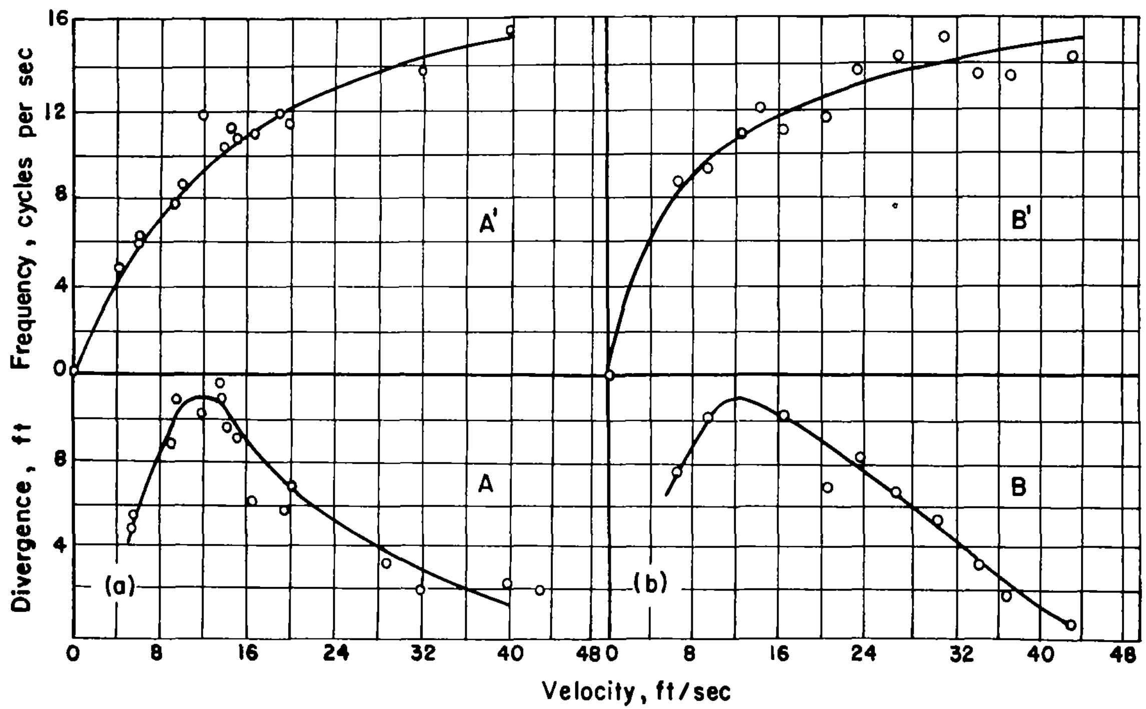


Figure 29.- Theoretical values of divergence, frequency and phase angle against velocity.



(a) Caster angle 5°

Caster length 0.43 cm

(b) Caster angle 20°

Caster length 1.73 cm

Figure 30.- Divergence and frequency of shimmy on belt machine compared with theory.

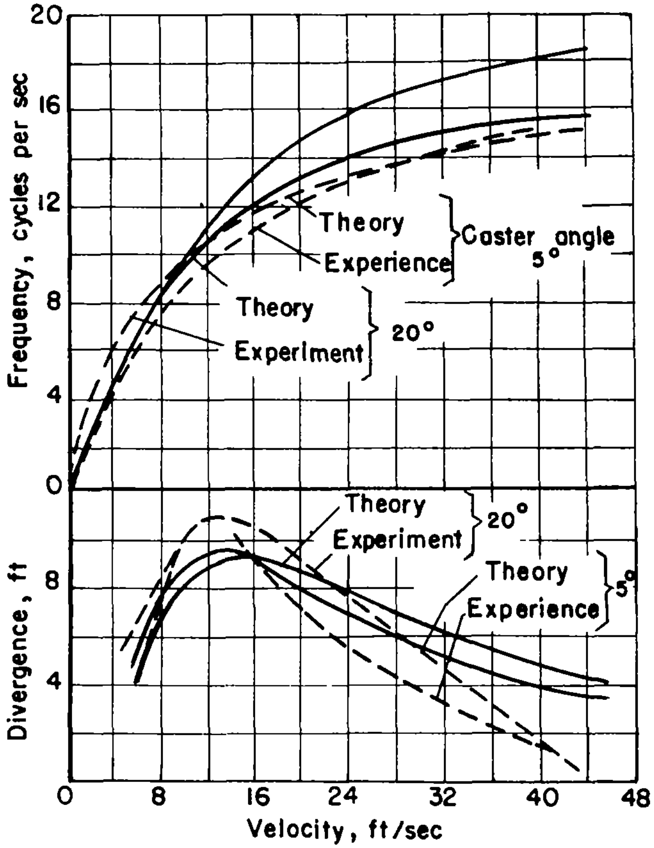


Figure 31.- Comparison of the experimental values of the divergence and the frequency of shimmy, observed on the belt machine, and the calculated values.

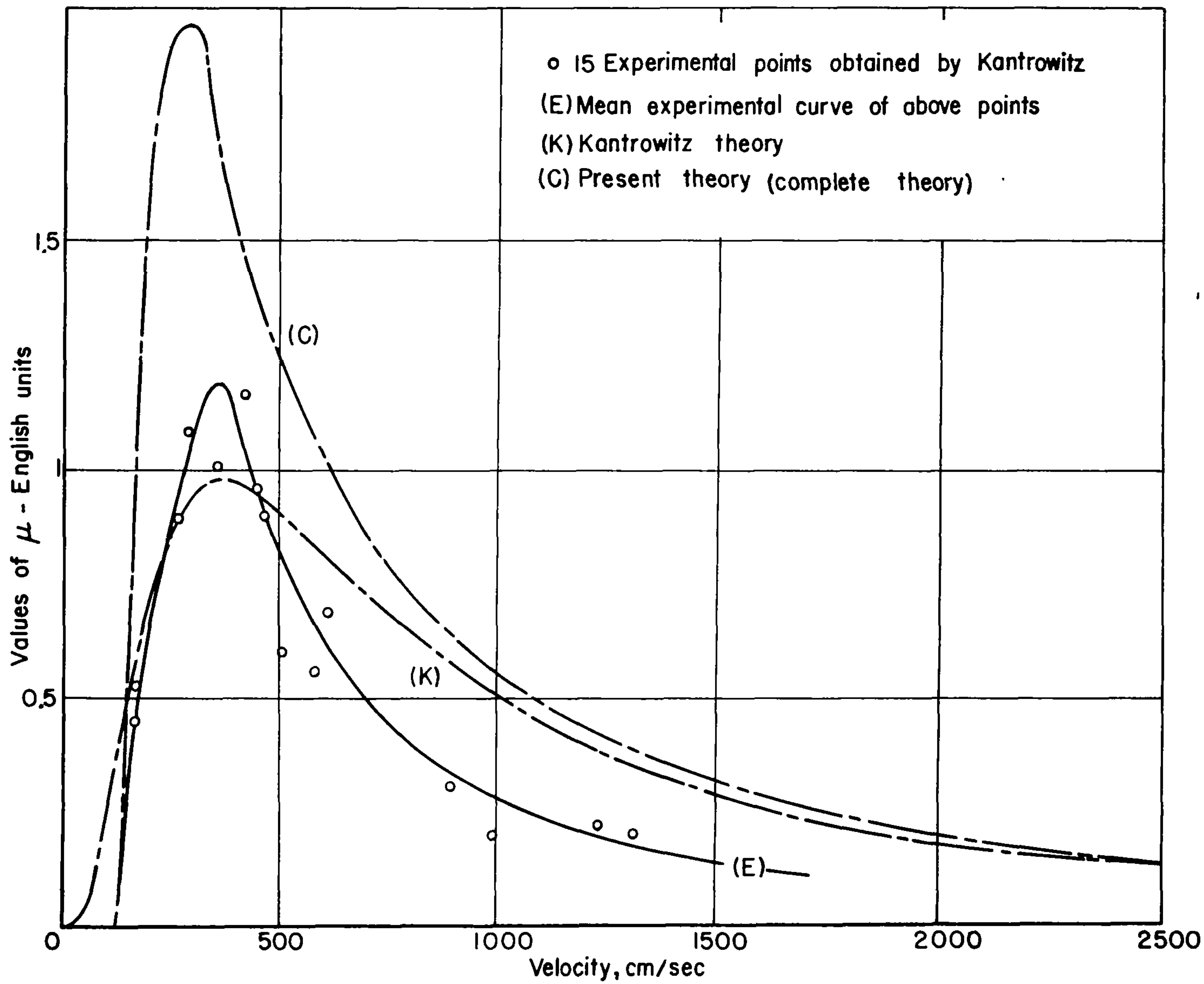

o 15 Experimental points obtained by Kantrowitz
(E) Mean experimental curve of above points
(K) Kantrowitz theory
(C) Present theory (complete theory)
(C)
(K)
(E)
Values of μ - English units
1.5
1
0.5
0
500
1000
1500
2000
2500
Velocity, cm/sec

Figure 32

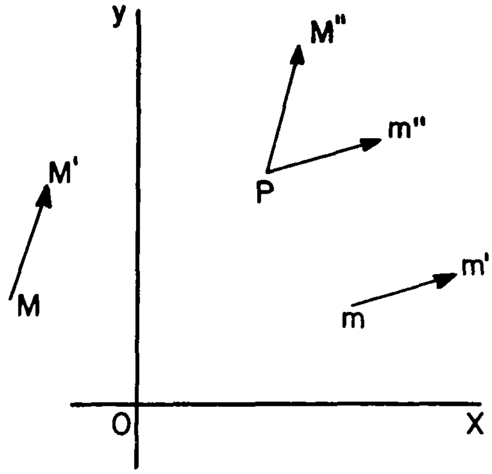


Figure 33

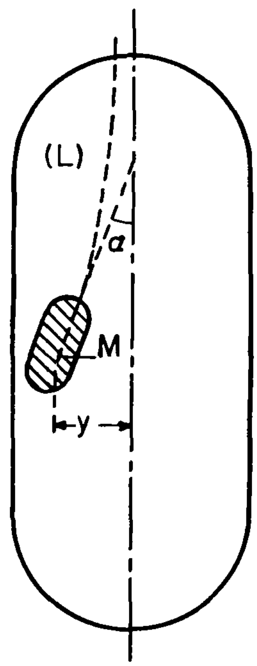
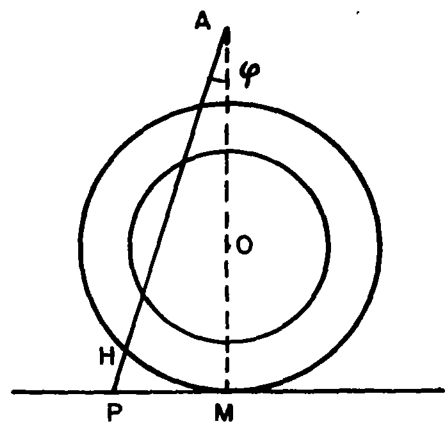


Figure 34



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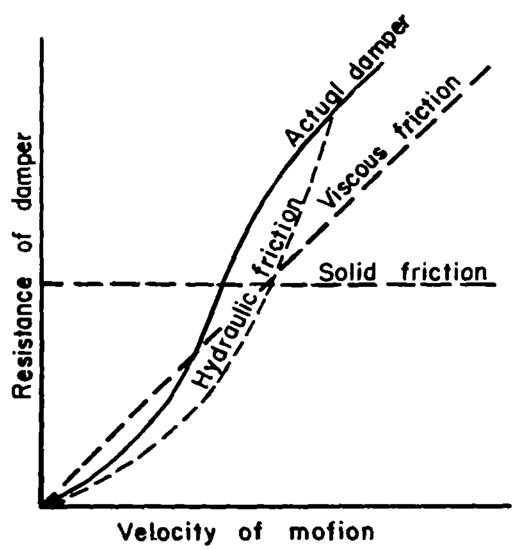

Figure 36

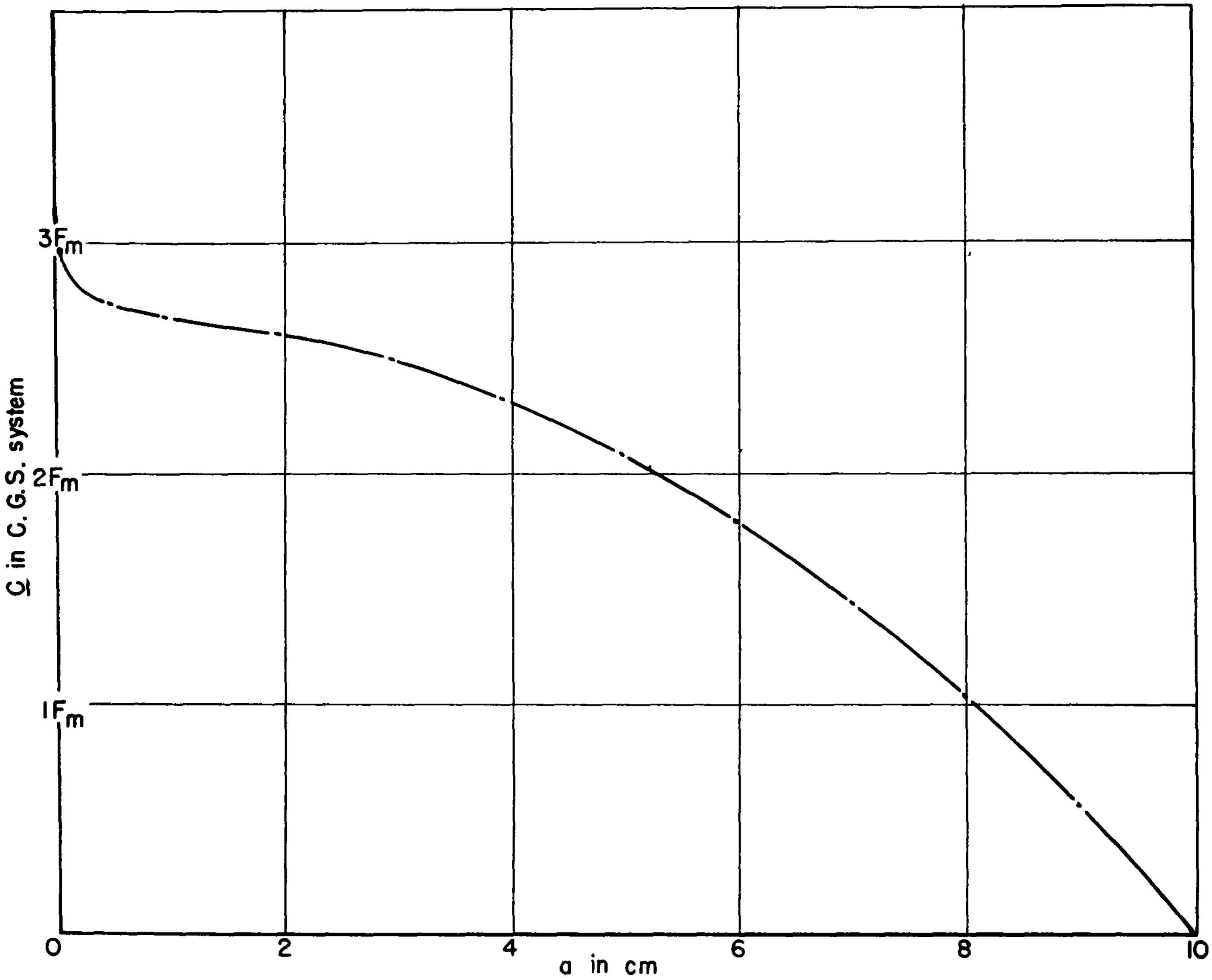

Figure 37

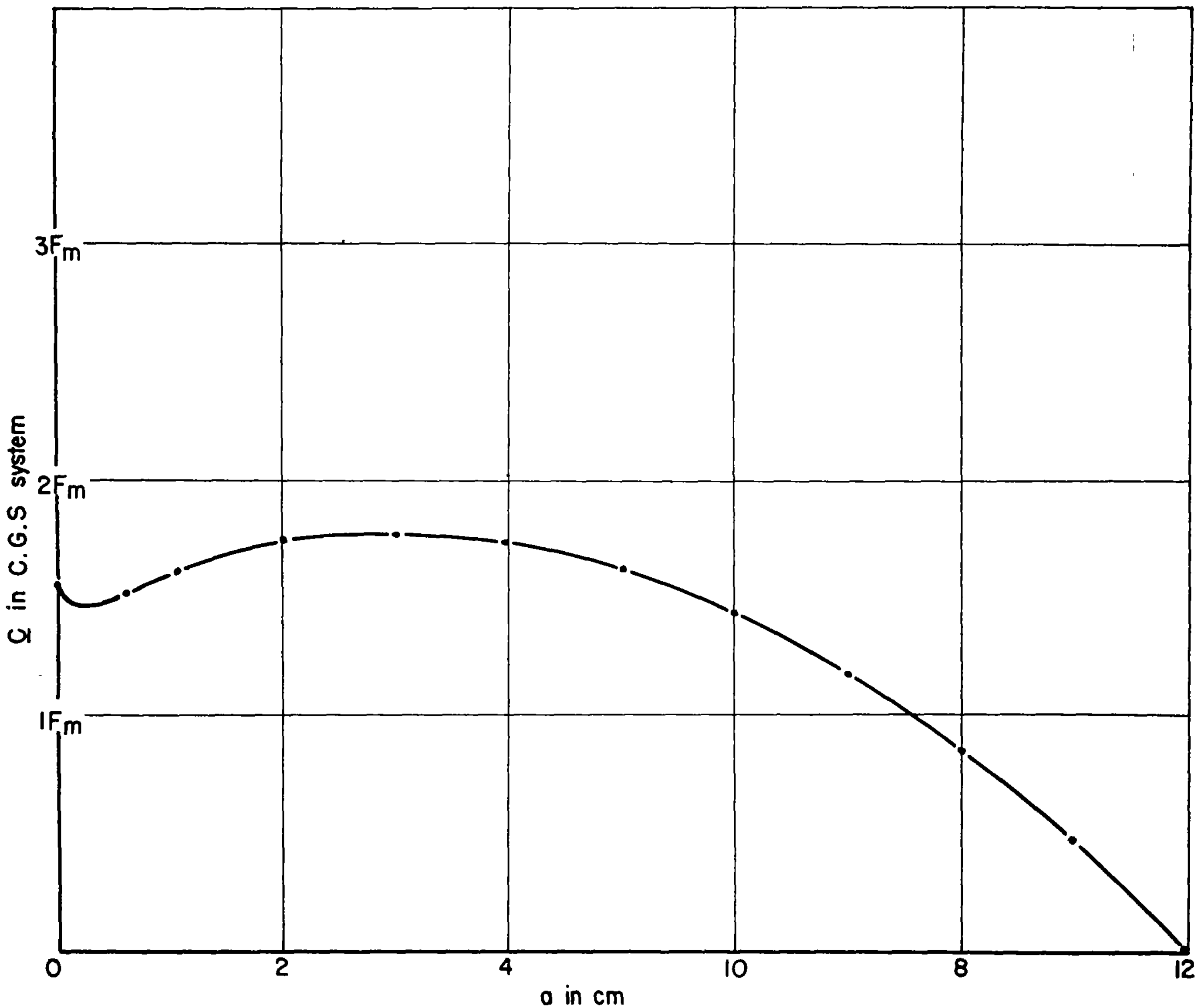

Figure 38

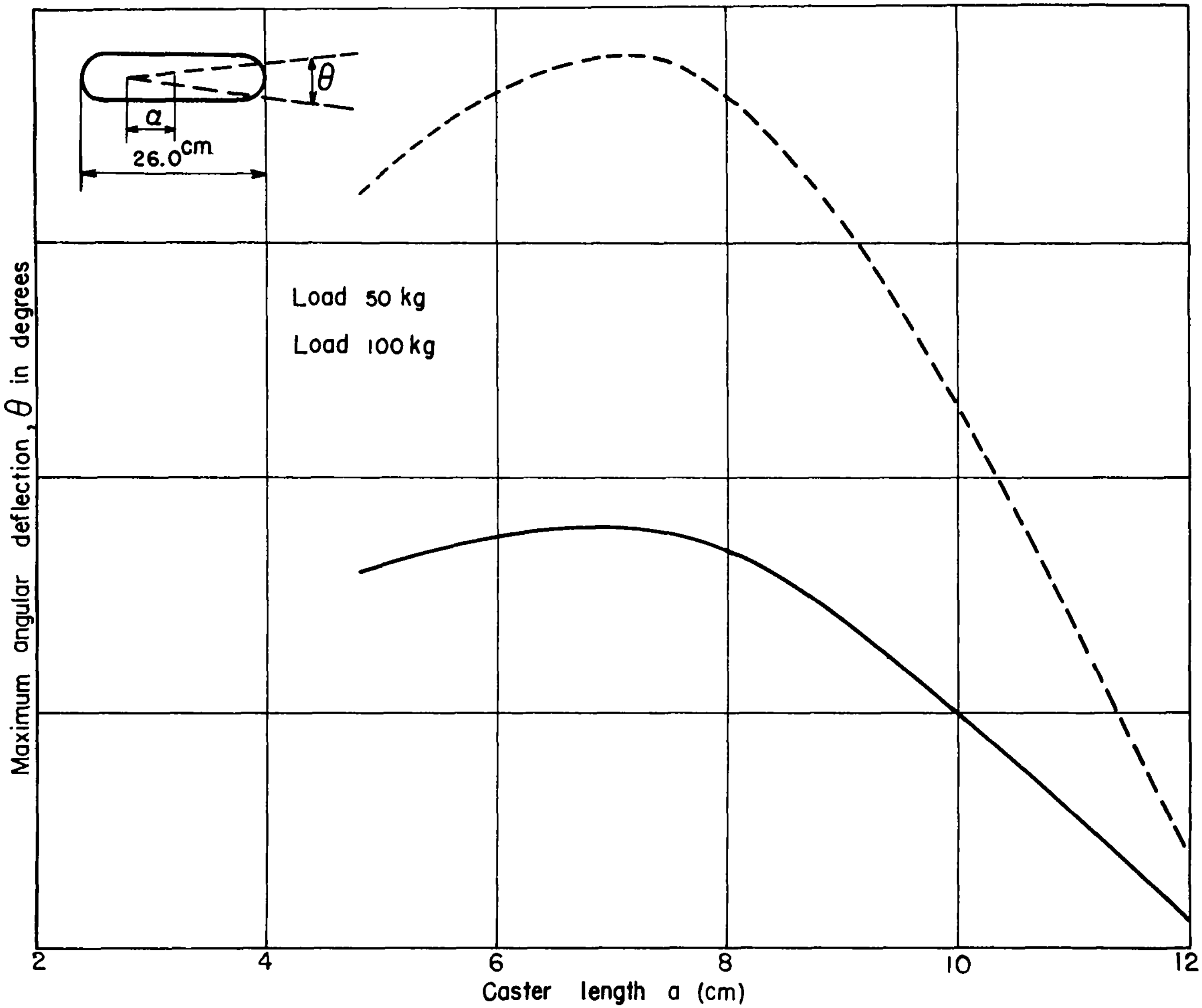
θ
a
26.0cm
Load 50 kg
Load 100 kg
Maximum angular deflection, θ in degrees
2
4
6
8
10
12
Caster length a (cm)

Figure 39

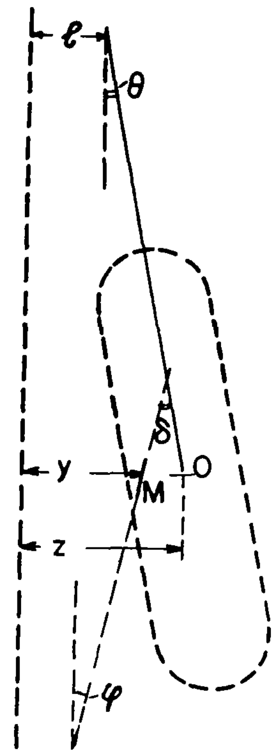

Figure 40

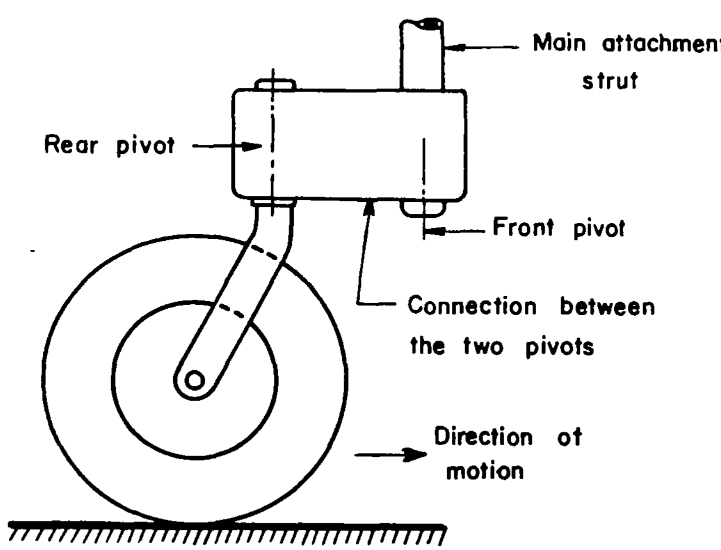

Figure 41

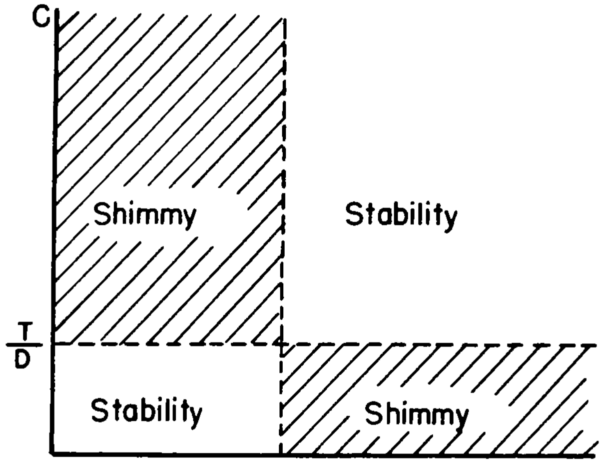

Figure 42

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